Systems, Sinusoids, the Fourier Transform, and Filters

David M.W. Evans[1], 19 August 2015

Abstract

This document presents the frequency-domain knowledge used in the notch-delay solar theory [Evans, The Notch-Delay Solar Hypothesis, 2016]: linear invariant systems, sinusoids, the Fourier transform, simple low pass, delay and notch filters, transfer functions and step responses, etc. Only information necessary to the theory is presented here. It culminates in developing the formulae for the range of possible step responses for the system from the Sun's total solar irradiance to the Earth's surface temperature, given the empirical observation of a notch filter in the frequency domain.

Administration

Cite as: Evans, David M.W., "*Systems, Sinusoids, the Fourier Transform, and Filters*", sciencespeak.com, 2015.

Download: pdf

The spreadsheet *climate.xlsm* by the same author applies the filters described here to climate datasets, contains most of the diagrams below, and contains the code and interface for checking calculations with numerical integration or FFT-approximations. For context, see the notch-delay solar theory webpage

If reading this as a pdf, consider navigating the document by headings—click on the "Bookmarks" icon in the small column on the left of the pdf window.

If reading this in Microsoft Word, consider navigating the document by headings in the Navigation Pane—on the "View" tab, "Show" section, click "Navigation Pane". Best at 120%.

1 Introduction

Analysis of systems using sinusoids was of fundamental importance in the technological progress of the last two centuries.

Around 1800 a young officer in Napoleon's army named Joseph Fourier founded a branch of mathematics called Fourier analysis, in which functions of time are expressed as sums of sinusoidal waves (he was studying heat propagation at the time). Analysis using sinusoids allowed us to understand linear invariant systems (LISs) for the first time. LISs are extremely

[1] david.evans@sciencespeak.com, http://sciencespeak.com/

common in nature, and the key to describing and unlocking their behavior is Fourier analysis—because sinusoids are rather special to LISs. One important application of Fourier analysis was in making sense of disparate phenomena like compasses and making frog's legs twitch with sparks, which culminated in 1861 in Maxwell's equations, the four equations that completely describe electromagnetism. Subsequently engineers and scientists could get down to exploiting electromagnetism, and we then had 150 years and counting of new wonders based on electricity and magnetism.

Linear invariant systems (LISs) are simple and ubiquitous systems that have a crucial property—if the input to a LIS is a sinusoid at a given frequency, then its output is also a sinusoid at the same frequency, though possibly with a different amplitude and phase. The universe is chock full of systems that are good approximations to LISs, especially in anything to do with electricity and magnetism. Many of the successes in physics and engineering from 1800 are based on Fourier analysis—an awful high proportion of modern technology wouldn't exist without this branch of mathematics. Some of the basic ideas of Fourier analysis have seeped into our technological consciousness, even of the non-technologically minded, so some of what follows in this introduction will be familiar to nearly all readers.

To get to the notch-delay solar theory from the datasets of solar irradiance and surface temperature data requires some understanding of systems and the frequency domain, which is the purpose of this document.

A "system" is anything with an input and an output, which is too broad a definition to be of much use. Add the conditions of linearity and invariance however, both fairly weak and common conditions, and the LIS is specific enough to form a useful construction of wide applicability. These two conditions single out one class of functions, the sinusoids, as special to the analysis of LISs.

Consider for example free space, or space with not much in it, like the atmosphere. It is a LIS for functions whose values are electric and magnetic field values. We are all accustomed to the ramifications of this, because we are familiar with the concepts of visible light, radio waves, UV, infrared, x-rays, microwaves, and so on—all of which are electromagnetic sinusoids at different frequencies. We implicitly analyze the fluctuations in the electric and magnetic fields around us into sinusoids at different frequencies, at least conceptually, because it is useful to think of them this way:

- Sinusoids at different frequencies don't interact with one another. For example light waves have no effect on radio waves—shining a torch on a radio doesn't interfere with its reception of a radio station.
- Sinusoids don't change frequencies as they go the through free space, air, or indeed many other things that are LISs. For example, radio waves are still radio waves when they pass through air or walls and so on, and don't suddenly become light waves or x-rays. You don't even need to retune your radio as you walk with it into a building.

The electric and magnetic fields could be analyzed into square waves or "waves" of some other shape—but it wouldn't be as useful, because the resulting "waves" would always be hopping between “frequencies” (or however you characterize these non-sinusoidal waves), and they would be forever interacting with each other in a myriad of ways. *Sinusoidal* waves are much simpler and more useful. The electric and magnetic fields would still be the same no matter how you think about them, of course, but using waves other than sinusoids would be a silly and confusing way of analyzing their fluctuations.

We are accustomed to the idea of the electromagnetic spectrum. At the highest frequencies are gamma rays and x-rays, and as we descend through the frequencies we come to UV, visible light, infrared, microwaves, and finally radio waves. The lowest frequency radio waves in use today are around 100,000 cycles per second. But what if you keep going to even lower frequencies? The same rules about LISs apply, but now we can talk about waves at one cycle per second, or one cycle per year, or one cycle per 11 years or per thousand years. These last few frequencies are the sinusoids of interest for the climate. The Sun emits these frequencies, which we perceive as gradual fluctuations in solar radiation, and technically they are part of the electromagnetic spectrum too. Just as infrared doesn’t interact with visible light, sinusoids at one cycle per year do not interact cycles at 11 cycles per year in free space, and so on. They’re the same phenomena mathematically, just on a different scale.

In the notch-delay solar theory we turn this branch of mathematics onto the relationship between total solar irradiance (TSI) and the surface temperatures on Earth.

Consider the system whose input is TSI and whose output is the surface temperature. It is presumably invariant, because its properties are unlikely to change much with time. For small perturbations of temperature, such as those over the last few thousand years, it is presumably linear—nearly all systems are linear for sufficiently small perturbations, and the climate system is widely assumed to be linear for such perturbations. So the system would appear to be a LIS, so Fourier methods are applicable.

The only measured data of interest on the Sun that goes back more than a few decades is the count of sunspots. The sunspot record, which starts in 1610 AD, has been converted to TSI using models based on the observed relationship between sunspot numbers and TSI over the last few decades (we have only been able to measure the tiny change in TSIs from 1978, with satellites; before that the TSI was called “the solar constant”). So if we are looking for a solar link to global warming on a climatic time scale, the sunspot record, or a TSI reconstruction from the sunspots, is about the only source of information we have on what the Sun has been doing.

This document starts with systems and deduces frequency domain behavior, focusing on notch, delay, and low pass filters.

A low pass filter mimics the thermal momentum of the climate system, simply smoothing out the impact of changes in heating and cooling from the Sun in accordance with a simple and obvious differential equation. Basically changes in incoming energy must accumulate over time to change the temperature.

A notch filter mimics the new and remarkable empirical observation that the frequencies of the sunspot cycle are greatly attenuated in the terrestrial surface temperature. A notch filter with the observed amplitudes at the various frequencies can be either causal or non-casual, which opens up an intriguing possibility. If non-causal, the response of the filter *precedes* its corresponding stimulus, which is impossible, so the notch filter would have to be accompanied by a delay in order for it to be physically realizable—which suggests the possibility of a delay of the order of the notch period. The notch period, corresponding to the average length of the sunspot cycle, is ~11 years. Or the notch filter could be causal. In any case, a delay of ~11 years between changes in TSI and changes in global surface temperature has been observed several times in disparate works, but apparently mostly interpreted by the researchers as delays in the propagation of heat around the Earth (though the magnitude of the warmings is much greater than the direct warming effect of the changes in TSI).

The notch-delay hypothesis proposes a hitherto unknown force from the Sun, called force X, that warms the Earth by affecting its albedo—how much sunlight is reflected back out to space by the clouds and ice etc. without heating the Earth. (Yes, "force X" sounds cartoonish, but the inspiration for the cartoons are x-rays, which were so named by Wilhelm Röntgen when their cause and nature were unknown.) Force X is lower when the Sun flips the polarity of its magnetic field, which it does every ~11 years as part of the full solar cycle (~22 years). The times when force X is lowest exactly coincide with the times when the TSI peaks during the solar cycle. The observed notching—the prominent peaks in TSI are *not* found in the Earth's surface temperature record—is because as warming from TSI peaks, the warming from force X is in a trough. They cancel, roughly. The two are in exact synchronicity through the irregular solar "cycle"; the notching could only be caused by a solar phenomenon.

The *full* solar cycle (called the Hale cycle) is ~22 years on average, which tends to get overlooked because most solar phenomena are proportional to the *square* of the Sun's magnetic field (which repeats about every ~11 years). The proposed delay of ~11 years—partly observed, partly suggested by the causality of notch filters, partly deduced by fitting the TSI and temperature data—between changes in TSI and changes in force X suggests that force X lags half a full solar cycle (180°) behind the TSI. The TSI is the bulk radiation coming from the Sun, but the composition of that radiation, particularly in UV and extreme UV, changes over the cycle. The notch-delay hypothesis proposes that TSI serves as a leading indicator of force X and thus changes in Earth's surface temperature, because changes in TSI precede changes in force X by about one sunspot cycle (half a full solar cycle).

This delay of ~11 years could explain, for instance, why global temperatures kept rising until about 1998 or so after the TSI stopped rising around 1986. Indeed, without the delay, it is difficult to see how changes in the TSI could be the major influence on surface temperatures.

A note on housekeeping: The special functions I (indicator), sgn (signum), step, eta, and pha (phase) are used sporadically in this document; they are defined in Appendix A.

2 System Definitions

A system is anything with an input and an output each describable by a function, while a linear invariant system (LIS) is a system that is both linear and invariant.

2.1 System

A system is an entity with an input function and an output function. If only functions of time t are of interest, then a system is anything whose input is a function of time and whose output is a function of time. Thus, a system maps a function to a function (where as a function maps a value to a value). If the input function to a system S is g_{IN}, then the output function is denoted by $S\{g_{\mathrm{IN}}\}$.

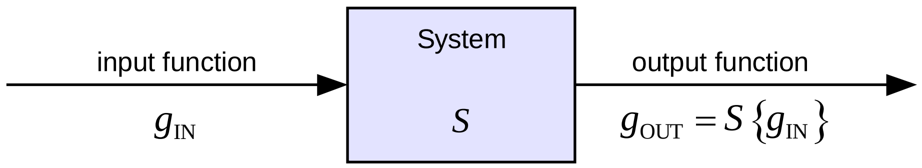


Figure 1: A system.

2.2 Linear System

A system is scalar if and only if

$$S\{ag\} = aS\{g\} \tag{1}$$

for all input functions g and all real numbers a. A system is superpositioning if and only if

$$S\{g_1 + g_2\} = S\{g_1\} + S\{g_2\} \tag{2}$$

for all input functions g_1 and g_2. A system is linear if and only if it is both scalar and superpositioning, that is, if and only if

$$S\{ag_1 + bg_2\} = aS\{g_1\} + bS\{g_2\} \tag{3}$$

for all input functions g_1 and g_2 and for all real numbers a and b. Thus, if the input to a linear system is a linear combination of input functions, the system effectively handles each function in the combination separately, as if the other functions in the combination were not present or had no effect.

2.3 Invariant System

A system is invariant if and only if

$$S\{g_{\mathrm{IN}}\} = g_{\mathrm{OUT}} \quad \Rightarrow \quad S\{t \mapsto g_{\mathrm{IN}}(t-\lambda)\} = [t \mapsto g_{\mathrm{OUT}}(t-\lambda)] \tag{4}$$

for all input functions g_{IN} and real numbers λ. Thus a system is time-invariant if time-shifting the input causes the output to be time-shifted by the same amount. More simply, a system is time-invariant if its properties do not change with time.

Here we have used the notation that a function g can also be denoted by "$t \mapsto g(t)$", which means that g maps the argument t to the value $g(t)$. By the way, it is common in system diagrams and in equations describing systems to omit the "$t \mapsto$" as understood and to just write a function as “$g(t)$”, which leaves the diagram or equation looking as if the system is mapping values when it is really mapping functions.

2.4 Linear Invariant System (LIS)

A linear invariant system (LIS) is a system that is both linear and invariant. Thus a LIS is scalar, superpositioning, and does not change its properties. A LIS is also called a filter, especially when its role is seen as shaping the spectrum of a signal passing through the system.

3 Impulse Responses

The impulse response of a system is its output function when the input function is an “impulse”, a function that is zero everywhere except when its argument is zero, and with one unit of input (that is, the area under the input function is one). It is the basic theoretical tool that allows the output of a LIS to be calculated from its input.

3.1 Impulse

An impulse is defined as the delta function δ, a special “function” whose value is zero everywhere except at zero, but which when present in an integral behaves as if the area under it is one when its argument is zero:

$$\int_{-\infty}^{\infty} g(t)\delta(t)\,dt = \int_{0^+}^{0^+} g(t)\delta(t)\,dt = \int_{0}^{0} g(t)\delta(t)\,dt = g(0) \tag{5}$$

for any real-valued function g defined on the real numbers $\mathbb{R}$. (There is no actual function that can fulfill this last condition, but we pretend there is, perhaps thinking of δ as the limit of a series of impulse-like functions appropriate to the given situation. This awkwardness has more to do with overcoming shortcomings with integration than because δ is physically unreal. In nearly any practical physical context, a series of impulse-like functions is readily discernible.)

The shifted delta function $t \mapsto \delta(t-\lambda)$ is zero for all values of t except λ, so integration with it “picks out” the value of a function at λ:

$$\int_{-\infty}^{\infty} g(t)\delta(t-\lambda)\,dt = \int_{\lambda^-}^{\lambda^+} g(t)\delta(t-\lambda)\,dt = \int_{\lambda}^{\lambda} g(t)\delta(t-\lambda)\,dt = g(\lambda) \tag{6}$$

for any function g defined on the real numbers. Significantly, the delta function allows us to express any function as a linear combination of impulses:

$$g(t) = \int_{-\infty}^{\infty} g(u)\delta(t-u)\,du\,. \tag{7}$$

3.2 Impulse Response

The impulse response h of a system S is the output function when the input is a delta function:

$$h = S\{\delta\} = S\{t \mapsto \delta(t)\}\,. \tag{8}$$

3.3 Calculating the Output of a LIS

Let the input to a LIS S be any function g_{IN}. Then the output of S can be expressed as a linear combination of shifted impulse responses:

$$\begin{aligned} S\left\{t \mapsto g_{\text{IN}}(t)\right\} &= S\left\{t \mapsto \int_{-\infty}^{\infty} g_{\text{IN}}(u)\delta(t-u)\,du\right\} \\ &= \int_{-\infty}^{\infty} g_{\text{IN}}(u) S\left\{t \mapsto \delta(t-u)\right\} du \\ &= \int_{-\infty}^{\infty} g_{\text{IN}}(u)\left[t \mapsto h(t-u)\right] du, \end{aligned} \tag{9}$$

where the equalities are respectively by Eq. (7), linearity, and invariance. Thus the value of the output function at t is a linear combination of impulse responses:

$$g_{\text{OUT}}(t) = \int_{-\infty}^{\infty} g_{\text{IN}}(u)h(t-u)du = \int_{-\infty}^{\infty} g_{\text{IN}}(t-v)h(v)dv \tag{10}$$

where we have made the substitution $v = t-u$ to get the final expression. The integrals in this equation are called convolution integrals, and g_{OUT} is said to be the convolution of g_{IN} and h, written as $g_{\text{IN}} * h$:

$$g_{\text{OUT}} = g_{\text{IN}} * h, \quad \text{or} \quad g_{\text{OUT}}(t) = \left\{g_{\text{IN}} * h\right\}(t), \;\; t \in \mathbb{R}. \tag{11}$$

4 Sinusoids

Sinusoids play a special role with LISs. Here we first define them, then we derive that special relationship. Apart from some obvious uses in trigonometry, the significance of sinusoids is limited to their special role in analyzing LISs.

4.1 Definition

A function is a sinusoid (or is sinusoidal) in some real variable t if it is a cosine or sine function in t, of the form

$$t \mapsto A\cos\left(2\pi ft - \phi\right), \;\; t \in \mathbb{R}.$$

The three parameters of the sinusoid are its:

- Frequency f, in cycles per unit of t. The period of the sinusoid is $1/f$.
- Amplitude A.
- Phase ϕ (in radians).

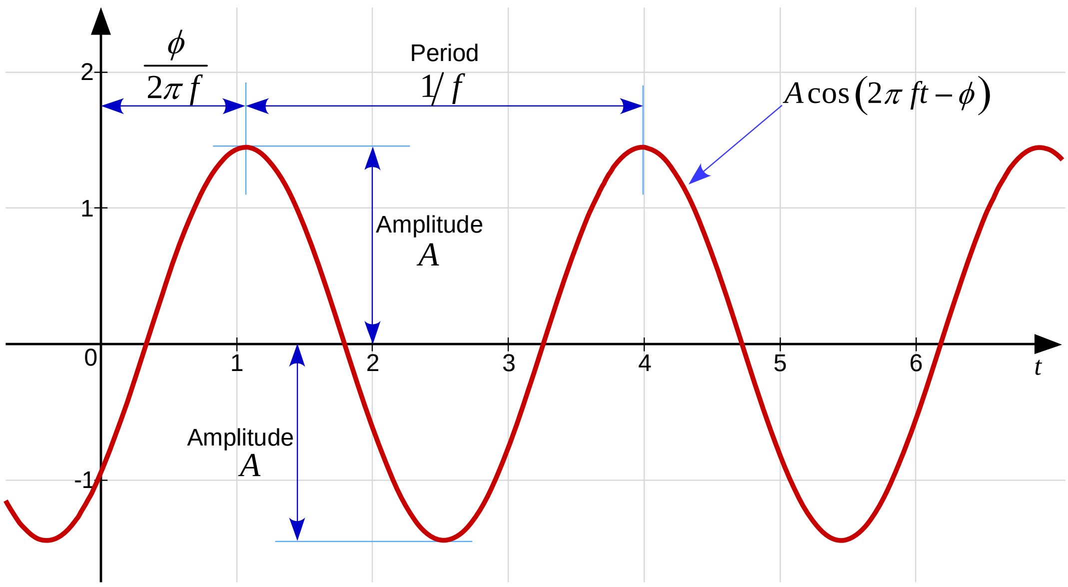


Figure 2: The sinusoid in *t* at frequency *f*, with amplitude *A* and phase *φ*.

4.2 Polar and Rectangular Coordinates

The sinusoid $t \mapsto A\cos(2\pi ft - \phi)$ is said to be expressed in polar coordinates, because it is expressed in terms of an amplitude *A* and phase ϕ. By basic trigonometry

$$A\cos(2\pi ft - \phi) = A\cos(\phi)\cos(2\pi ft) + A\sin(\phi)\sin(2\pi ft).$$

The cosine and sine coefficients, or rectangular coordinates, of this sinusoid are $A\cos(\phi)$ and $A\sin(\phi)$ respectively.

The sinusoid $t \mapsto B_C \cos(2\pi ft) + B_S \sin(2\pi ft)$ is said to be expressed in rectangular coordinates, because it is expressed in terms of a cosine coefficient B_C and a sine coefficient B_S. By basic trigonometry and the phase function (Appendix A.5),

$$B_C \cos(2\pi ft) + B_S \sin(2\pi ft) = \sqrt{B_C^2 + B_S^2}\cos(2\pi ft - \text{pha}(B_C, B_S)).$$

The amplitude and phase, or polar coordinates, of the sinusoid are thus $\sqrt{B_C^2 + B_S^2}$ and $\text{pha}(B_C, B_S)$ respectively.

4.3 Sinusoids and all LISs

Consider an arbitrary LIS *S* whose input and outputs are functions of *t*, and whose impulse response is *h*. Let the input function be an arbitrary sinusoid in time *t* at frequency f_0, with amplitude *A* and phase ϕ, namely

$$t \mapsto g_{IN}(t) = A\cos(2\pi f_0 t - \phi). \tag{12}$$

By Eq. (10), the value of the output function at *t* is

$$
\begin{aligned}
g_{\text{OUT}}(t) &= \int_{-\infty}^{\infty} g_{\text{IN}}(t-v)h(v)\,dv \\
&= \int_{-\infty}^{\infty} A\cos\left[2\pi f_0(t-v)-\phi\right]h(v)\,dv \\
&= \int_{-\infty}^{\infty} A\left[\cos\left(2\pi f_0 t-\phi\right)\cos\left(2\pi f_0 v\right)+\sin\left(2\pi f_0 t-\phi\right)\sin\left(2\pi f_0 v\right)\right]h(v)\,dv \\
&= A\cos\left(2\pi f_0 t-\phi\right)\int_{-\infty}^{\infty}\cos\left(2\pi f_0 v\right)h(v)\,dv \\
&\quad +A\sin\left(2\pi f_0 t-\phi\right)\int_{-\infty}^{\infty}\sin\left(2\pi f_0 v\right)h(v)\,dv.
\end{aligned} \tag{13}
$$

The integrals in the last expression, namely

$$
\begin{aligned}
C &= \int_{-\infty}^{\infty}\cos\left(2\pi f_0 v\right)h(v)\,dv \\
S &= \int_{-\infty}^{\infty}\sin\left(2\pi f_0 v\right)h(v)\,dv,
\end{aligned} \tag{14}
$$

evaluate to numbers that are independent of *t*, so they are constants. Thus

$$
g_{\text{OUT}}(t) = AC\cos\left(2\pi f_0 t-\phi\right)+AS\sin\left(2\pi f_0 t-\phi\right). \tag{15}
$$

Convert this sinusoid to polar coordinates by letting

$$
\begin{aligned}
C &= \alpha\cos\theta, \quad & \alpha &= \sqrt{C^2+S^2} \\
S &= \alpha\sin\theta, \quad & \theta &= \text{pha}(C,S).
\end{aligned} \tag{16}
$$

Hence

$$
\begin{aligned}
g_{\text{OUT}}(t) &= A\alpha\cos\left(\theta\right)\cos\left(2\pi f_0 t-\phi\right)+A\alpha\sin\left(\theta\right)\sin\left(2\pi f_0 t-\phi\right) \\
&= A\alpha\cos\left(2\pi f_0 t-\left[\phi+\theta\right]\right).
\end{aligned} \tag{17}
$$

From this we can draw several notable conclusions:

1. The output function is also a sinusoid in *t* at frequency f_0.
2. The amplitude of the output sinusoid is equal to the amplitude of the input sinusoid multiplied by α.
3. The phase of the output sinusoid is equal to the phase of the input sinusoid plus θ.
4. α and θ are independent of the amplitude and phase of the input sinusoid, so they apply to *all* input sinusoids at frequency f_0.
5. The behavior of the LIS for input sinusoids of frequency *f* can be characterized by just two real numbers, an amplitude multiplier α and a phase addend θ.

If the input to any LIS is a sinusoid at frequency f_0, then its output is also a sinusoid at frequency f_0 (thus, the sinusoids at frequency f_0 are eigenfunctions of all linear invariant systems). The output sinusoid may have a different amplitude and phase to the input sinusoid, but it is guaranteed to be a sinusoid with frequency f_0. Furthermore, a given LIS always "transfers" a sinusoid at a given frequency from the input to the output with the *same* amplitude amplification and *same* phase shift (hence the "transfer function" terminology, below).

In general, if the input to a LIS is a non-sinusoidal function (that remains finite as its argument becomes infinite) then the output is not guaranteed to be any particular function. For example, if the input function to an arbitrary LIS is a square wave then the output function tends to be something like a square wave but with the corners rounded off—so it is no longer a square wave. The sinusoids, however, are special.

By the linearity of a LIS, if the input function of a LIS is a sum of sinusoids, each at a different frequency, then the output function, by Eq. (17), is also a sum of sinusoids at the same frequencies, where the relationship between the output and input sinusoids at any given frequency is determined by just the amplitude amplification and phase shift of the LIS at that frequency. So for input functions that are sums of sinusoids, we can calculate the output function just from knowing the how the LIS changes amplitudes and phases at each frequency. The Fourier transform (section 5, below) shows that many functions of physical interest can be expressed as just such a sum of sinusoids, one at each frequency—so we can calculate the output function of a LIS just from its amplitude- and phase-change properties.

Given the ubiquity of LISs, this explains why sinusoids are of great interest to science and technology. The very general notion of a system, with the mild constraints of linearity and invariance, turns out to be amenable to a simple analysis where the behavior of the system can be summarized merely by an amplitude and phase at each frequency.

4.4 Complex Numbers as an Accounting Tool for Sinusoids

Complex numbers can be viewed as an accounting system that might have been invented specifically for sinusoids and LISs (historically it wasn't, but that's just an unfortunate accident). In complex multiplication, amplitudes are multiplied and phases are added—just like the effect of a LIS on an input sinusoid as per Eq. (17).

Let us represent a sinusoid by the complex number that has the same amplitude and phase (it could hardly get any simpler than that):

$$A\cos\left(2\pi f_0 t-\phi\right) \longleftrightarrow Ae^{i\phi} = A\exp(i\phi) = A\cos\phi + i\,A\sin\phi \qquad (18)$$

The right hand side of this correspondence is a complex exponential, in polar coordinates as $Ae^{i\phi}$ and in rectangular coordinates as $A\cos\phi + i\,A\sin\phi$. The crucial ingredient is the imaginary number i, the square root of -1. Don't take the square root of -1 literally, because it doesn't exist; instead, think of i as merely combining two real numbers into a single entity, a "complex" number, with a property that is very useful in this context, namely, i^2 equals -1.

To continue section 4.3, let us represent the effect of the LIS at frequency f_0 by the complex number with the amplitude and phase of the changes it causes at that frequency, namely $\alpha e^{i\theta}$. The (complex) product of this with the complex number in Eq. (18) is

$$Ae^{i\phi}\alpha e^{i\theta} = A\alpha\exp\left[i(\phi+\theta)\right] \qquad (19)$$

(multiply the amplitudes and add the phases). By virtue of having the same amplitude and phase, this is the complex number that represents the output sinusoid calculated by Eq. (17) when the input sinusoid is that in Eq. (18). Thus the action of the LIS in transferring the input sinusoid in Eq. (12) to the output sinusoid in Eq. (17) at the same frequency can by represented by a complex multiplication. Similarly, adding complex numbers corresponds to adding sinusoids with the same frequency. So we can dispense with calculating explicitly with sinusoidal functions, and instead just calculate with the complex numbers that represent them—much simpler.

5 The Fourier Transform (FT)

The Fourier transform is a tool for analyzing a function of a continuous real variable (such as time) into a sum of sinusoids, called the spectrum of the function.

5.1 FTs of Complex-Valued Functions

Let g be a function defined on all real numbers (such as for all time). Let g be complex-valued (because complex numbers are an accounting tool for representing sinusoids, this is somewhat unmotivated and even nonsensical, but it is traditional). Let $g(t)$ and $g(-t)$ remain finite as t becomes infinite. Let g not be "extremely" discontinuous (or the integrals here do not converge; this is generally not an issue with "real-world" functions). Let the complex Fourier transform of g be the complex-valued function F. Let the argument of F vary over all the real numbers and be called the frequency f.

Synthesis:

$$g(t) = \int_{-\infty}^{\infty} F(f) \exp(i 2\pi ft) df \quad \text{for } t \in \mathbb{R}. \tag{20}$$

Analysis:

$$F(f) = \int_{-\infty}^{\infty} g(t) \exp(-i 2\pi ft) dt \quad \text{for } f \in \mathbb{R}. \tag{21}$$

We write the real and imaginary parts of F as F_{real} and F_{img} (which are real-valued):

$$F(f) = F_{\text{real}}(f) + iF_{\text{img}}(f). \tag{22}$$

The relationship between g and its (complex-valued) complex Fourier transform F can be expressed by the complex Fourier transform operator **F**:

$$\mathbf{F}\{g\} = f \mapsto F(f) \quad \text{or} \quad \mathbf{F}\{g\}(f) = F_{\text{real}}(f) + iF_{\text{img}}(f). \tag{23}$$

The Fourier transform synthesizes g as a sum of complex exponentials, typically

$$\exp(\pm i 2\pi ft) = \cos(2\pi ft) \pm i \sin(2\pi ft), \tag{24}$$

one at each real frequency f (though see Figure 2: a sinusoid with a negative frequency has the same period as a sinusoid with the absolute value of that frequency, which is ambiguous). Thus, after applying the complex multiplication in its integrand, the synthesis integral synthesizes g as a sum of sinusoids. The units of frequency are cycles per unit of t; for example, if t is measured in years then f is measured in cycles per year (cycles are dimensionless).

We can calculate F from g (by analysis, or the forward transform) and g from F (by synthesis, or the inverse transform), so the information in the function can be fully represented either as g (in which case we say it is in the time domain, if g is a function of time) or as F (in the frequency domain). The Fourier transform is thus invertible.

We haven't proved that, given the definition of the Fourier transform in the analysis Eq. (21), the Fourier transform synthesis in Eq. (20) is correct. There is an intricate mathematical proof, reasonably well-known, that we won't reproduce here.

5.2 FTs of Real-Valued Functions

Almost all functions of interest are real-valued, and the Fourier transform becomes simpler when g is real-valued. Everything above about complex-valued functions still applies, because a real-valued function is just a complex-valued function whose imaginary part is zero.

If g is real-valued then its Fourier transform is complex-valued, but by Eq. (21)

$$\left.\begin{aligned} F_{\text{real}}(-f) &= F_{\text{real}}(f) \\ F_{\text{img}}(-f) &= -F_{\text{img}}(f) \end{aligned}\right\} \quad f \ge 0, \tag{25}$$

so the values of the Fourier transform at negative frequencies are redundant.

It is easier to work with Fourier transforms of real-valued functions by focusing on their cosine and sine parts, denoted here by B_{C} and B_{S} respectively. (The "B" is for Professor Ronald Bracewell, late of Electrical Engineering at Stanford University, who played a large part in the modern revival of the Fourier transform, applied it in radio astronomy and image reconstruction, and wrote an influential text on Fourier transforms in 1978.) Further, we need only consider non-negative frequencies, because the values of the Fourier transform at negative frequencies give you no extra information about the spectrum of a real-valued function. These two policies remove the analysis of imaginary functions and redundant (aka aliased) frequencies from the picture, allowing us to focus just on the essentials without stumbling over irrelevant symmetries and unnecessary complications. Finally, we use the eta function η for taking care of the inevitable factors of two: η is one, except that it is zero when f is zero (Appendix A; η is the number of normal or non-edge frequencies). Now we can define the real Fourier transform (or Bracewell transform) of a real-valued function g:

Synthesis:

$$g(t) = \int_0^\infty \left[B_{\text{C}}(f)\cos(2\pi ft) + B_{\text{S}}(f)\sin(2\pi ft) \right] df \quad \text{for } t \in \mathbb{R}. \tag{26}$$

Analysis:

$$\left.\begin{aligned} B_{\text{C}}(f) &= 2^\eta \int_{-\infty}^\infty g(t)\cos(2\pi ft)\, dt \\ B_{\text{S}}(f) &= 2^\eta \int_{-\infty}^\infty g(t)\sin(2\pi ft)\, dt \end{aligned}\right\} \quad \text{for } f \ge 0. \tag{27}$$

The cosine and sine components B_{C} and B_{S} of the real Fourier transform are often combined into a complex number, giving a single analysis equation:

$$B(f)=B_{\mathrm{C}}(f)+iB_{\mathrm{S}}(f)=2^{\eta}\int_{-\infty}^{\infty}g(t)\exp\left(i\,2\pi ft\right)dt\,,\quad f\geq 0\,. \tag{28}$$

The synthesis equation (26) then becomes a dot product (that is, the sum of product of corresponding components):

$$g(t)=\int_{0}^{\infty}B(f)\bullet\exp\left(i\,2\pi ft\right)df\,,\quad t\in\mathbb{R}\,. \tag{29}$$

The dot product expands as in Eq. (26) in rectangular coordinates, while in polar coordinates

$$Ae^{i\phi}\bullet\exp\left(i\,2\pi ft\right)=A\cos\left(2\pi ft-\phi\right),\quad A,\phi\in\mathbb{R}\,. \tag{30}$$

The relationship between g and its (complex-valued) real Fourier transform B can be expressed by the real Fourier transform operator **B**:

$$\mathbf{B}\left\{g\right\}=f\mapsto B(f)\,,\quad\text{or}\quad\mathbf{B}\left\{g\right\}(f)=B(f)=2^{\eta}\int_{-\infty}^{\infty}g(t)\exp\left(i\,2\pi ft\right)dt\,. \tag{31}$$

For a real-valued function, the relationship between its complex Fourier transform and its real Fourier transform is

$$\left.\begin{aligned}B_{\mathrm{C}}(f)&=\ \ 2^{\eta}F_{\mathrm{real}}(f)\\ B_{\mathrm{S}}(f)&=-2^{\eta}F_{\mathrm{img}}(f)\end{aligned}\right\}\quad\text{for } f\geq 0\,, \tag{32}$$

or

$$B(f)=2^{\eta}F^{*}(f)\quad\text{for}\ \ f\geq 0 \tag{33}$$

where the asterisk superscript indicates the complex conjugate (which means change the sign of i). (The 2^{η} factor may be regarded as "folding" the negative part of the real number line representing frequency over onto the positive part. So the complex conjugate is an arbitrary sign change in the frequency in Eq. (21).)

The synthesis explicitly expresses g as a sum of sinusoids, one at each (non-negative) frequency (see Fig. 2; a sinusoid with a positive frequency has an unambiguous period).

5.3 FTs of Time Series

Real-world data typically comes as a series, sampled from an underlying function of a continuous variable (typically time, in which case the series is a "time series"). There are "discrete" versions of the Fourier transform to analyze such a time series into a sum of sinusoidal time series, which approximate the Fourier transform of the underlying function of a continuous variable—see [Evans, The Optimal Fourier Transform (OFT), 2013]. We are only concerned with functions of a continuous real variable in this document.

5.4 FTs of Derivatives and Integrals

The Fourier transforms of integrals and derivatives of a function are easily calculated from the Fourier transform of the function, which is invaluable for solving differential equations.

Let a real-valued function g have a real Fourier transform B, as per Eq. (28). Apply a derivative to the Fourier synthesis in Eq. (29):

$$\begin{aligned}\frac{d}{dt}g(t) &= \frac{d}{dt}\left\{\int_{-\infty}^{\infty}\left[B_{\mathrm{C}}(f)\cos(2\pi ft)+B_{\mathrm{S}}(f)\sin(2\pi ft)\right]df\right\}\\ &= \int_{-\infty}^{\infty}\left[B_{\mathrm{C}}(f)\frac{d}{dt}\cos(2\pi ft)+B_{\mathrm{S}}(f)\frac{d}{dt}\sin(2\pi ft)\right]df\\ &= \int_{-\infty}^{\infty}\left[2\pi fB_{\mathrm{S}}(f)\cos(2\pi ft)-2\pi fB_{\mathrm{C}}(f)\sin(2\pi ft)\right]df\,.\end{aligned}$$

The last expression is the FT synthesis integral for dg/dt, and $-i(B_{\mathrm{C}}+iB_{\mathrm{S}})$ equals $B_{\mathrm{S}}-iB_{\mathrm{C}}$, so the real Fourier transform of dg/dt is $-i\,2\pi fB(f)$. Thus

$$\mathbf{B}\left\{\frac{d^n g}{dt^n}\right\}(f)=(-i\,2\pi f)^n\mathbf{B}\{g\}(f)\,,\quad \text{for } n=0,1,2,\ldots,\ f\geq 0\,. \tag{34}$$

Now apply an integral to the Fourier synthesis:

$$\begin{aligned}\int g(t)\,dt &= \int\left[\int_{-\infty}^{\infty}\left[B_{\mathrm{C}}(f)\cos(2\pi ft)+B_{\mathrm{S}}(f)\sin(2\pi ft)\right]df\right]dt\\ &= \int_{-\infty}^{\infty}\left[\int B_{\mathrm{C}}(f)\cos(2\pi ft)\,dt+\int B_{\mathrm{S}}(f)\sin(2\pi ft)\,dt\right]df\\ &= \int_{-\infty}^{\infty}\left[-\frac{B_{\mathrm{S}}(f)}{2\pi f}\cos(2\pi ft)+\frac{B_{\mathrm{C}}(f)}{2\pi f}\sin(2\pi ft)\right]df\,.\end{aligned}$$

This last is the FT synthesis integral for $\int g(t)\,dt$, and $(-i)^{-1}(B_{\mathrm{C}}+iB_{\mathrm{S}})$ equals $i(B_{\mathrm{C}}+iB_{\mathrm{S}})$ which equals $-B_{\mathrm{S}}+iB_{\mathrm{C}}$, so the real Fourier transform of $\int g(t)\,dt$ is $(i\,2\pi f)^{-1}B(f)$. Thus

$$\mathbf{B}\left\{\int\ldots\int g(t_i)\,dt_1\ldots dt_n\right\}(f)=(-i\,2\pi f)^{-n}\mathbf{B}\{g\}(f)\,,\quad \text{for } n=0,1,2,\ldots,\ f\geq 0\,. \tag{35}$$

The Fourier transform expresses g as a sum of sinusoids. Each time we differentiate or integrate g then we differentiate or integrate all those sinusoids, which all shift one quarter cycle (from cosine to negative sine with differentiation, or cosine to sine with integration) and get scaled by $2\pi f$ (differentiation) or $(2\pi f)^{-1}$ (integration).

5.5 FTs of Delayed Functions

Consider the delayed function $t\mapsto g(t-d)$ for some delay d that is independent of t. Its real Fourier transform is

$$\begin{aligned}2^{\eta}\int_{-\infty}^{\infty}g(t-d)\exp(i\,2\pi ft)\,dt &= 2^{\eta}\int_{-\infty}^{\infty}g(u)\exp[i\,2\pi f(u+d)]\,du\\ &= \exp(i\,2\pi fd)\,2^{\eta}\int_{-\infty}^{\infty}g(u)\exp(i\,2\pi fu)\,du\\ &= \exp(i\,2\pi fd)\,B(f),\end{aligned}$$

where B is the real Fourier transform of g, and we made the substitution $u=t-d$. Thus

$$\mathbf{B}\{t \mapsto g(t-d)\}(f) = i2\pi fd\,\mathbf{B}\{g\}(f)\,. \tag{36}$$

5.6 LISs, and the Time and Frequency Domains

In physics and engineering it is common that the input and output of a system are connected by a linear differential equation, in which the input and output are functions of time and the derivatives are with respect to time. Because the equation is in terms of time, and contains no explicit mention of frequency, this sort of description and analysis is said to be "in the time domain".

A system described by a linear differential equation is linear. If in addition the equation's coefficients are independent of time (constant with respect to time), the system is also invariant and therefore a LIS.

If the system is a LIS then frequency domain methods are appropriate: replace each function of time with its equivalent sum of sinusoids, one sinusoid in time at each frequency, using Fourier analysis. Then we can analyze the effect of the LIS (or linear differential equation) at each frequency in isolation, because what happens at other frequencies has no effect. Calculate the output sinusoid at each frequency, then add them all together to form the output function. Such an analysis, using explicit frequencies, and where only the sinusoids are dependent on time, is said to take place "in the frequency domain". Often the sinusoids are not even explicitly stated, but are merely implicit in the analysis as written.

To move the description of the LIS from the time domain to the frequency domain, take the Fourier transform of all functions of time—with the analysis integral—which converts them to functions of frequency. To move back to the time domain, apply the inverse Fourier transform to the functions of frequency—with the synthesis integral, thereby adding the sinusoids.

It is easier to solve linear differential equations in the frequency domain than in the time domain, essentially because the derivatives or integrals of sinusoids are just other sinusoids at the same frequency. In areas like electrical engineering, where circuits are typically LISs described by elaborate linear differential equations, equations are routinely solved by moving them to the frequency domain using the Fourier transform (or the Laplace transform, a generalization that analyzes functions into sums of exponentially growing sinusoids). The sinusoids are represented as complex numbers, while differentiation and integration become multiplication and division by a complex variable—so the linear differential equations become polynomials, which are much simpler to solve.

5.7 Taking the FT of Both Sides of an Equation

Suppose an equation says two functions of the same real variable t are equal, either

$$g_1 = g_2\,, \quad \text{or} \quad g_1(t) = g_2(t) \ \text{ for } t \in \mathbb{R}\,. \tag{37}$$

To "take the Fourier transform of both sides" means taking the Fourier transform of the function on each side and equating their values at each frequency:

$$\mathbf{B}\{g_1\}(f) = \mathbf{B}\{g_2\}(f)\,, \quad f \ge 0\,. \tag{38}$$

6 Transfer Functions

The transfer function of a LIS consists of the amplitude amplifications and phase shifts that the LIS causes at each frequency, encoded as a Fourier transform. It is also the Fourier transform of the impulse function (to within a scaling by a factor of two).

6.1 Transfer Function of a LIS

Consider an arbitrary LIS *S* whose input and outputs are functions of *t*, and whose impulse response is *h*, as in section 4.3. The (real) Fourier transform of the input sinusoid in Eq. (12), namely $t \mapsto A\cos\left(2\pi f_0 t - \phi\right)$, is

$$\mathbf{B}\{g_{\text{IN}}\}(f) = Ae^{i\phi}\delta(f - f_0), \quad f \geq 0, \tag{39}$$

because

$$\left.\begin{aligned} B_{\text{C}}(f) &= A\cos(\phi)\delta(f - f_0) \\ B_{\text{S}}(f) &= A\sin(\phi)\delta(f - f_0) \end{aligned}\right\} \quad \text{for } f \geq 0. \tag{40}$$

The output function is then the one in Eq. (17), whose Fourier transform is, similarly,

$$\mathbf{B}\{g_{\text{OUT}}\}(f) = A\alpha e^{i(\phi+\theta)}\delta(f - f_0), \quad f \geq 0. \tag{41}$$

The behavior of the LIS at frequency f_0 is determined entirely by α and θ, so it is completely captured by the (complex) ratio of the values of the Fourier transforms of the output to the input:

$$\frac{\mathbf{B}\{g_{\text{OUT}}\}(f_0)}{\mathbf{B}\{g_{\text{IN}}\}(f_0)} = \frac{A\alpha e^{i(\phi+\theta)}}{Ae^{i\phi}} = \alpha e^{i\theta}. \tag{42}$$

Notice that this (complex) value is independent of the amplitude and phase of the input sinusoid, so it applies to all non-zero input sinusoids. The value is specific to the frequency f_0, so we can construct a function of frequency out of the values in Eq. (42) as f_0 varies. Accordingly, we define the transfer function of the LIS *S* as

$$H(f) = \frac{\mathbf{B}\{g_{\text{OUT}}\}(f)}{\mathbf{B}\{g_{\text{IN}}\}(f)}, \quad f \geq 0, \tag{43}$$

for any input function g_{IN} whose sinusoid at each frequency is non-zero (to avoid zeroes in the denominator in Eq. (42)). The value of the LIS's transfer function at any frequency f_0 tells us how the LIS "transfers" the input sinusoid to the output at f_0. Because any function of interest is the sum of one sinusoid at each frequency (Eq. (26)), the transfer function completely characterizes the LIS.

With the transfer function, for any input function g_{IN} we can compute the spectrum of the output function g_{OUT}:

$$\mathbf{B}\{g_{\text{OUT}}\}(f) = H(f)\mathbf{B}\{g_{\text{IN}}\}(f), \quad f \geq 0. \tag{44}$$

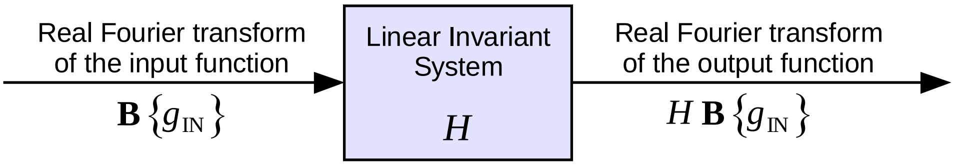


Figure 3: A LIS in the frequency domain. *H* is the transfer function of the LIS. The real Fourier transforms and the transfer function are each complex-valued functions of non-negative frequency.

Whereas a linear differential equation is the usual method of describing a system in the time domain, a transfer function is the usual description of a system in the frequency domain.

6.2 Transfer Function and Impulse Response

The transfer function H from the previous sub-section is a complex-valued function of frequency so it is a Fourier transform, but of what? Consider the Fourier transform of the LIS's impulse function h. Its cosine and sine components at frequency f_0 are, comparing Eq.s (27) and (14),

$$\left.\begin{aligned} H_{\mathrm{C}}(f_0) &= 2^{\eta(f_0)} C \\ H_{\mathrm{S}}(f_0) &= 2^{\eta(f_0)} S \end{aligned}\right\} \quad \text{for } f_0 \geq 0\,, \tag{45}$$

so the real Fourier transform of h is

$$\mathbf{B}\{h\}(f) = 2^{\eta} H(f)\,, \quad f \geq 0\,. \tag{46}$$

(With the complex Fourier transform, the 2^{η} frequency-folding factor is not needed because frequencies may be negative: $\mathbf{F}\{h\}(f) = H(f)$, $f \in \mathbb{R}$.)

6.3 Convolution

Taking the Fourier transform of both sides of the convolution in Eq. (10),

$$\begin{aligned} \mathbf{B}\{g_{\mathrm{IN}} * h\}(f) &= \mathbf{B}\left\{\int_{-\infty}^{\infty} g_{\mathrm{IN}}(u) h(t-u)\,du\right\}(f) \\ &= 2^{\eta}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_{\mathrm{IN}}(u) h(t-u)\,du\right]\exp\left(i\,2\pi ft\right)dt \\ &= 2^{\eta}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_{\mathrm{IN}}(u) h(t-u)\exp\left(i\,2\pi ft\right)dt\right]du \\ &= 2^{\eta}\int_{-\infty}^{\infty} g_{\mathrm{IN}}(u)\exp\left(i\,2\pi fu\right)\left[\int_{-\infty}^{\infty} h(t-u)\exp\left(i\,2\pi f(t-u)\right)dt\right]du \\ &= 2^{\eta}\int_{-\infty}^{\infty} g_{\mathrm{IN}}(u)\exp\left(i\,2\pi fu\right)\left[2^{-\eta}\mathbf{B}\{h\}(f)\right]du \\ &= 2^{-\eta}\mathbf{B}\{h\}(f)\mathbf{B}\{g_{\mathrm{IN}}\}(f)\,, \end{aligned} \tag{47}$$

assuming g_{IN} and h are sufficiently well-behaved to swap the order of the integrals. Thus, summarizing Eq.s (10), (44), and (46) in the frequency domain,

$$\mathbf{B}\{g_{\mathrm{OUT}}\}(f) = \mathbf{B}\{g_{\mathrm{IN}} * h\}(f) = 2^{-\eta}\mathbf{B}\{h\}(f)\mathbf{B}\{g_{\mathrm{IN}}\}(f) = H(f)\mathbf{B}\{g_{\mathrm{IN}}\}(f)\,, \quad f \geq 0\,, \tag{48}$$

or, where the real Fourier transforms of g_{IN} and g_{OUT} are G_{IN} and G_{OUT},

$$G_{\mathrm{OUT}}(f) = \mathbf{B}\{g_{\mathrm{IN}} * h\}(f) = 2^{-\eta}\mathbf{B}\{h\}G_{\mathrm{IN}}(f) = H(f)G_{\mathrm{IN}}(f)\,, \quad f \geq 0\,. \tag{49}$$

More generally, convolution in the time domain corresponds to (complex) multiplication in the frequency domain:

$$\mathbf{B}\{g_1 * g_2\}(f) = \mathbf{B}\left\{\int_{-\infty}^{\infty} g_1(u) g_2(t-u)\,du\right\}(f) = 2^{-\eta} G_1(f) G_2(f), \quad f \geq 0, \tag{50}$$

for any well-behaved functions g_1 and g_2 with real Fourier transforms G_1 and G_2. The result is the same, without the frequency-folding factor, if complex FTs are used instead:

$$\mathbf{F}\{g_1 * g_2\}(f) = \mathbf{F}\left\{\int_{-\infty}^{\infty} g_1(u) g_2(t-u)\,du\right\}(f) = F_1(f) F_2(f), \quad f \in \mathbb{R}, \tag{51}$$

where the complex Fourier transforms of g_1 and g_2 are F_1 and F_2.

6.4 Cascaded LISs

Two systems are in cascade if the output from one is the input to the other. The transfer function of the cascade is equal to the product of the transfer functions of the individual systems. Complex multiplication is commutative and associative, so the order of the individual systems within the combined system makes no difference to the transfer function of the combined system.

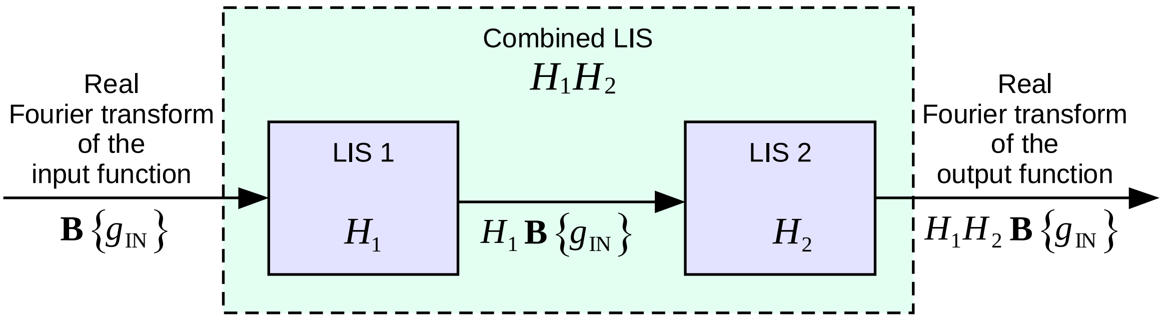


Figure 4: Two LISs in cascade, in the frequency domain. Each LIS is marked with its transfer function. The transfer function of the cascade of the two systems (in either order) is $H_1 H_2$.

7 Step Responses

The step response of a LIS is its output function when its input is a step function (see Appendix A.3), that is, if the input steps up from zero to one at time 0. The step response of a LIS is usually more intuitive and easier to verify experimentally than an impulse response, and is useful for understanding and comparing LISs. The output of a system can be computed from its step response and the input function.

7.1 Step Response

The (unit) step response of a system S is the output function when the input function is a unit step function, namely

$$r = S\{\text{step}\} = S\{t \mapsto \text{step}(t)\}.$$

The value of any input function g_{IN} at any time t can be expressed as a linear combination of unit steps:

$$g_{\text{IN}}(t) = \int_{-\infty}^{t} dg_{\text{IN}} = \int_{-\infty}^{\infty} \frac{dg_{\text{IN}}}{du} \text{step}(t-u)\, du\,. \tag{52}$$

If S is linear and invariant, then the output function of S is thus

$$\begin{aligned} S\{t \mapsto g_{\text{IN}}(t)\} &= S\left\{t \mapsto \int_{-\infty}^{\infty} \frac{dg_{\text{IN}}}{du} \text{step}(t-u)\, du\right\} \\ &= \int_{-\infty}^{\infty} \frac{dg_{\text{IN}}}{du} S\{t \mapsto \text{step}(t-u)\}\, du \\ &= \int_{-\infty}^{\infty} \frac{dg_{\text{IN}}}{du} [t \mapsto r(t-u)]\, du\,, \end{aligned}$$

where the first line is by Eq. (52), the second by linearity, and the third by invariance. Thus the value of the output function at time t is the same linear combination of step responses:

$$g_{\text{OUT}}(t) = \int_{-\infty}^{\infty} \frac{dg_{\text{IN}}}{du} r(t-u)\, du\,. \tag{53}$$

7.2 Causality

A causal system is one where the variable of the input and output functions is time and whose step response is zero for all times before the step in the input occurs. That is, a system is causal if the effect comes *after* the cause. Non-causal systems are not physically realizable.

By Eq. (53), the output of a causal system depends on past and current inputs but not on future inputs.

7.3 Calculating the Step Response from the Transfer Function

To calculate the step response r of a LIS from its transfer function H, apply Eq. (44) with the input function g_{IN} as the step function:

$$r = \mathbf{B}^{-1}\{H\,\mathbf{B}\{\text{step}\}\} \tag{54}$$

(assuming the output of the LIS is real-valued when the input is a step function.) To see this in more detail, first compute the real Fourier transform of the unit step function:

$$\mathbf{B}\{\text{step}\} = \frac{\delta(f)}{2} + \frac{i}{\pi f}, \quad f \ge 0\,. \tag{55}$$

(To confirm this, apply the synthesis integral in Eq. (26):

$$\int_0^\infty \left[B_\mathrm{C}(f)\cos(2\pi ft) + B_\mathrm{S}(f)\sin(2\pi ft) \right] df = \int_0^\infty \left[\frac{\delta(f)}{2}\cos(2\pi ft) + \frac{\sin(2\pi ft)}{\pi f} \right] df$$
$$= \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$
$$= \mathrm{step}(t), \quad t \in \mathbb{R}, \tag{56}$$

by the definite integral [Gradshteyn & Ryzhik, p. 405: 3.721#1 & pp.xliii].) Second, letting the transfer function of the LIS be expressed as

$$H(f) = H_\mathrm{Real}(f) + iH_\mathrm{Img}(f), \quad f \geq 0, \tag{57}$$

where H_Real and H_Img are real-valued, the real Fourier transform of the step response is

$$H\,\mathbf{B}\{\mathrm{step}\}(f) = \left[\frac{H_\mathrm{Real}(f)\delta(f)}{2} - \frac{H_\mathrm{Img}(f)}{\pi f} \right] + i\left[\frac{H_\mathrm{Img}(f)\delta(f)}{2} + \frac{H_\mathrm{Real}(f)}{\pi f} \right], \quad f \geq 0. \tag{58}$$

Third, the step response of the LIS is

$$r(t) = \mathbf{B}^{-1}\left\{ H\,\mathbf{B}\{\mathrm{step}\} \right\}(t)$$
$$= \int_0^\infty \left\{ \begin{array}{l} \left[\dfrac{H_\mathrm{Real}(f)\delta(f)}{2} - \dfrac{H_\mathrm{Img}(f)}{\pi f} \right]\cos(2\pi ft) \\ + \left[\dfrac{H_\mathrm{Img}(f)\delta(f)}{2} + \dfrac{H_\mathrm{Real}(f)}{\pi f} \right]\sin(2\pi ft) \end{array} \right\} df$$
$$= \frac{H_\mathrm{Real}(0)}{2} + \int_0^\infty \frac{H_\mathrm{Real}(f)\sin(2\pi ft) - H_\mathrm{Img}(f)\cos(2\pi ft)}{\pi f} df, \quad t \in \mathbb{R}. \tag{59}$$

Note that this becomes Eq. (56) for the identity LIS, for which *H* is unity for all frequencies. Note also that, because $\cos(2\pi ft)/\pi f$ is increasing without limit, $H_\mathrm{Img}(f)$ must approach zero fast as *f* approaches zero from above. Finally, note that if $H_\mathrm{Img}(f)$ is non-zero then the LIS changes the phase of the sinusoids at *f*, and it will contribute a term to the integral that is an even function in *t* (symmetric around zero *t*)—thus, unless the phase changes caused by the LIS are carefully arranged to cancel for negative *t* in the integral, the LIS will be non-causal.

Alternatively, in polar coordinates:

$$\mathbf{B}\{\mathrm{step}\} = \frac{\delta(f)}{2} + \frac{\exp(i\pi/2)}{\pi f}, \quad f \geq 0 \tag{60}$$

$$H(f) = A_H(f)\exp\left[i\phi_H(f) \right], \quad f \geq 0 \tag{61}$$

$$H\,\mathbf{B}\{\mathrm{step}\}(f) = \frac{\delta(f)A_H(f)\exp\left[i\phi_H(f) \right]}{2} + \frac{A_H(f)}{\pi f}\exp\left\{ i\left[\phi_H(f) + \pi/2 \right] \right\}, \quad f \geq 0 \tag{62}$$

$$r(t) = \frac{A_H(0)}{2}\cos\left[\phi_H(0) \right] + \int_0^\infty \frac{A_H(f)}{\pi f}\sin\left[2\pi ft - \phi_H(f) \right] df, \quad t \in \mathbb{R}. \tag{63}$$

8 Low Pass Filters (LPFs)

Consider the following differential equation, which describes a LIS that arises in a wide variety of physical contexts:

$$\frac{1}{2\pi f_{\mathrm{B}}} \frac{d}{dt} g_{\mathrm{OUT}} = w g_{\mathrm{IN}} - g_{\mathrm{OUT}} \tag{64}$$

where g_{IN} and g_{OUT} are functions of a continuous variable t, while f_{B} and w are constants (independent of t), and $f_{\mathrm{B}} > 0$. Taking the real Fourier transform of both sides and applying Eq. (34),

$$\frac{-i\, 2\pi f}{2\pi f_{\mathrm{B}}} \mathbf{B}\{g_{\mathrm{OUT}}\}(f) = w\mathbf{B}\{g_{\mathrm{IN}}\}(f) - \mathbf{B}\{g_{\mathrm{OUT}}\}(f), \quad f \geq 0, \tag{65}$$

so the transfer function of the system is

$$H_{\mathrm{LPF}}(f) = \frac{\mathbf{B}\{g_{\mathrm{OUT}}\}(f)}{\mathbf{B}\{g_{\mathrm{IN}}\}(f)} = \frac{w}{1 - i\left(f / f_{\mathrm{B}}\right)}, \quad f \geq 0. \tag{66}$$

This is the transfer function of a first order low pass filter (that is, with one pole), the simplest type of low pass filter, such as an RC filter in electronics. See Fig. 5.

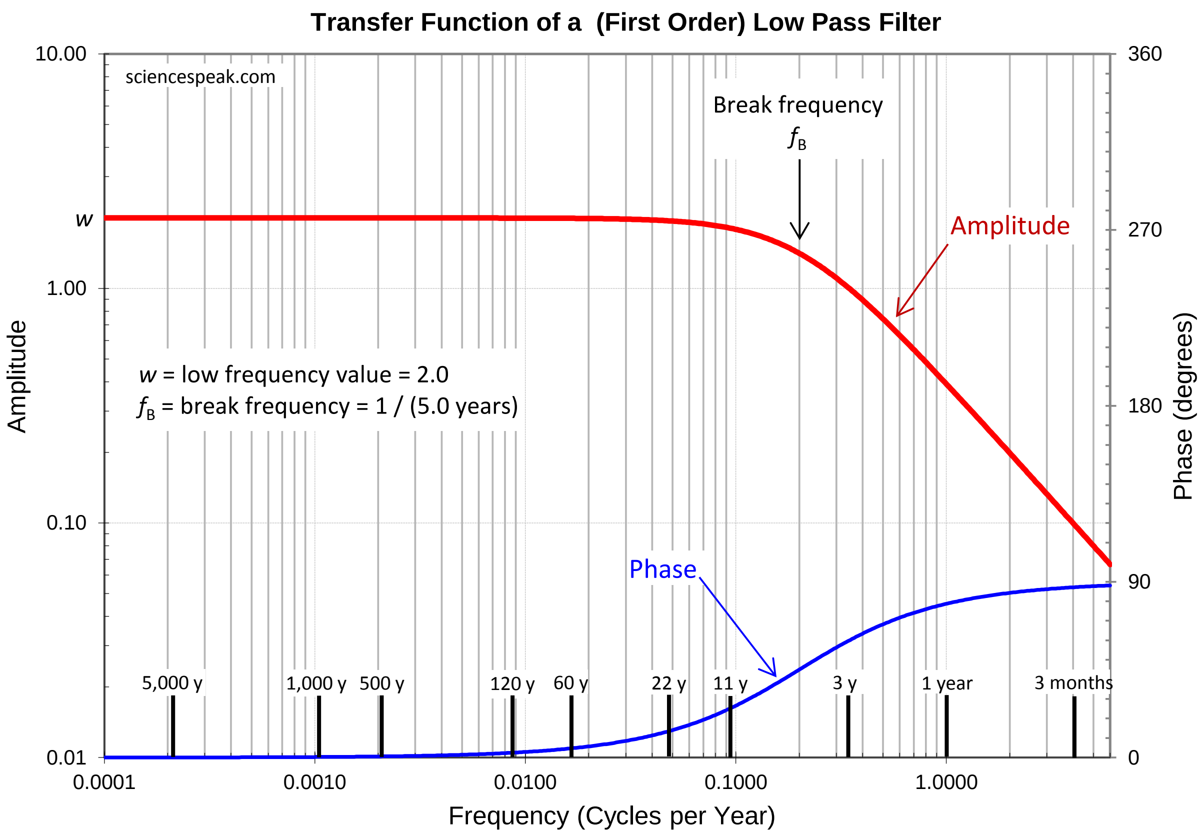


Figure 5: Transfer function of a low pass filter. Note the logarithmic scales on the axes. A low pass filter "passes" sinusoids with frequencies below f_B and "blocks" those above (the higher the frequency, the more it is blocked).

At low frequencies

$$H_{\text{LPF}}(f) \simeq w \quad \text{for } f \ll f_{\text{B}}, \tag{67}$$

so all the sinusoids in the input appear in the output amplified by the factor *k*. In the limit as the frequency goes to zero the derivatives with respect to *t* go to zero (because everything changes only very slowly), in which case Eq. (64) simply says that g_{OUT} is equal to wg_{IN}, which agrees with Eq. (67). *w* is the low frequency value. At high frequencies

$$H_{\text{LPF}}(f) \simeq \frac{iw}{(f/f_{\text{B}})} \quad \text{for } f \gg f_{\text{B}}, \tag{68}$$

so each sinusoid in the input is attenuated by wg_{B}/f and lagged by 90° (for example, cosine becomes sine) as it makes its way to the output. Because low-frequency sinusoids pass through unattenuated while high frequency sinusoids are not passed, this LIS is known as a (first order) low pass filter. The behavior switches from passing to not passing centered on the frequency f_{B}, which is known as the break frequency. The effect of a low pass filter is to smooth a function: the high frequency sinusoids provide the sharply-changing features or sharp corners of a function, and it is these that are most attenuated.

To compute the step response we first reveal the real and imaginary parts of the transfer function:

$$H_{\text{LPF}}(f) = \frac{w\left[1+i\left(f/f_{\text{B}}\right)\right]}{1+\left(f/f_{\text{B}}\right)^2} = \frac{wf_{\text{B}}}{f^2+f_{\text{B}}^2}\left(f_{\text{B}}+if\right), \quad f \geq 0. \tag{69}$$

Then by Eq. (59) the step response is

$$\begin{aligned} r_{\text{LPF}}(t) &= \frac{H_{\text{LPF,Real}}(0)}{2} + \int_0^\infty \frac{H_{\text{LPF,Real}}(f)\sin(2\pi ft) - H_{\text{LPF,Img}}(f)\cos(2\pi ft)}{\pi f}\,df \\ &= \frac{w}{2} + wf_{\text{B}}\int_0^\infty \left[\frac{f_{\text{B}}\sin(2\pi ft)}{\pi f\left(f^2+f_{\text{B}}^2\right)} - \frac{\cos(2\pi ft)}{\pi\left(f^2+f_{\text{B}}^2\right)}\right]df \\ &= \frac{w}{2} + kw\,\text{sgn}(t)\frac{1-\exp\left(-2\pi|t|f_{\text{B}}\right)}{2} - w\frac{\exp\left(-2\pi|t|f_{\text{B}}\right)}{2} \\ &= \frac{w}{2}[1+\text{sgn}(t)] - w\frac{1+\text{sgn}(t)}{2}\exp\left(-2\pi|t|f_{\text{B}}\right) \\ &= w\,\text{step}(t) - w\,\text{step}(t)\exp\left(-2\pi|t|f_{\text{B}}\right) \\ &= w\left[1-\exp\left(-2\pi f_{\text{B}}t\right)\right]\text{step}(t), \end{aligned} \tag{70}$$

by the definite integrals [Gradshteyn & Ryzhik, p. 408: 3.725#1 & 3.723#2]. Thus the step response is just a step with the corner at zero *t* smoothed off, and more smoothing when f_{B} is smaller (that is, fewer higher frequency sinusoids are passed). The step response is zero for *t* less than 0, so the low pass filter is causal. See Fig. 6.

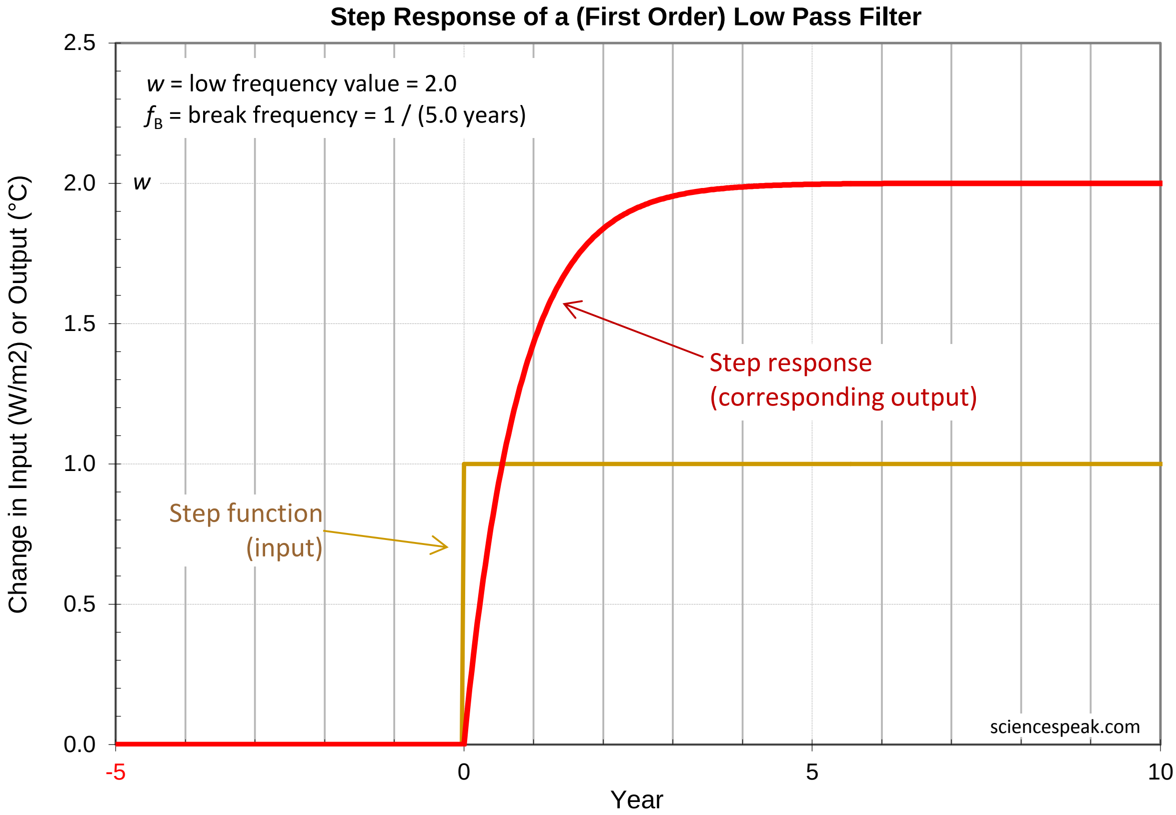


Figure 6: Step response of a low pass filter (the one in Fig. 5). A low pass filter smooths off sharp corners.

The impulse response of the low pass filter is, by Eq. (46),

$$\begin{aligned} h_{\mathrm{LPF}}(t) &= \mathbf{B}^{-1}\left\{2^{\eta} H_{\mathrm{LPF}}(f)\right\}(t) \\ &= \int_0^{\infty} 2^{\eta}\left[H_{\mathrm{LPF,real}}(f)\cos(2\pi ft) + H_{\mathrm{LPF,imag}}(f)\sin(2\pi ft)\right]df \\ &= 2wf_{\mathrm{B}}\int_0^{\infty}\left[\frac{f_{\mathrm{B}}\cos(2\pi ft)}{f^2+f_{\mathrm{B}}^2} + \frac{f\sin(2\pi ft)}{f^2+f_{\mathrm{B}}^2}\right]df \\ &= 2wf_{\mathrm{B}}\left[\frac{\pi}{2}\exp\left(-2\pi f_{\mathrm{B}}|t|\right) + \frac{\pi}{2}\exp\left(-2\pi f_{\mathrm{B}}|t|\right)\mathrm{sgn}(t)\right] \\ &= 2\pi f_{\mathrm{B}} w\exp\left(-2\pi f_{\mathrm{B}} t\right)\mathrm{step}(t), \end{aligned} \tag{71}$$

by the definite integrals [Gradshteyn & Ryzhik, pp. 406: 3.723#2,#3].

9 Delay Filters

Consider a LIS whose output g_{OUT} is simply a delayed version of the input g_{IN} (the naming of the filter assumes both are functions of time), for which

$$g_{\mathrm{OUT}}(t) = g_{\mathrm{IN}}(t-d), \quad t \in \mathbb{R}, \tag{72}$$

where the delay d is any real number. Taking the Fourier transform of both sides and applying Eq. (36) gives the transfer function as

$$H_{\mathrm{Delay}}(f) = \frac{\mathbf{B}\{g_{\mathrm{OUT}}\}(f)}{\mathbf{B}\{g_{\mathrm{IN}}\}(f)} = \exp(i\,2\pi\, fd), \quad f \geq 0 \,. \tag{73}$$

The amplitude of H_{Delay} is always unity, but the phases are modified in proportion to the frequency and the delay. See Fig. 7.

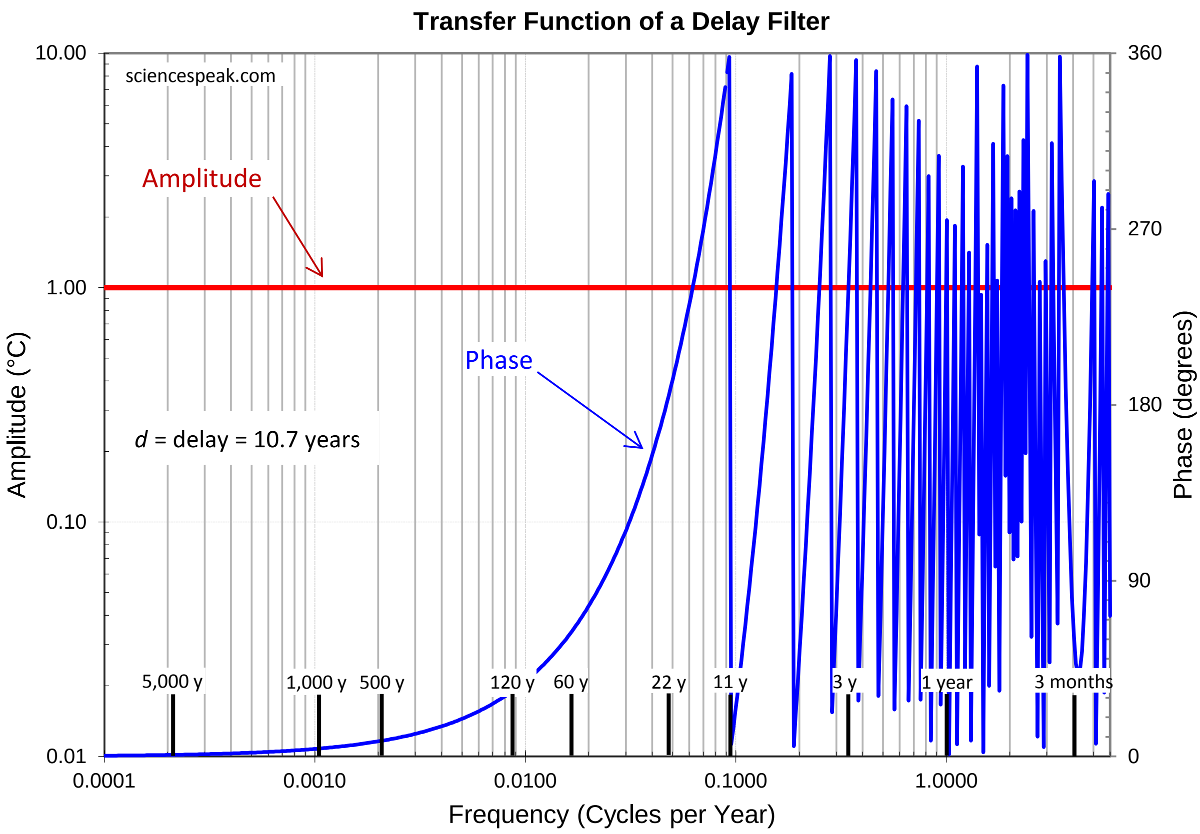


Figure 7: The transfer function of a delay filter. The amplitude is unity at al frequencies, but the phase change accelerates as frequency increases (the graphics fail to keep up; it should show the phase "wrapping around").

To compute the step function, first split the transfer function into its real and imaginary parts:

$$H_{\mathrm{Delay}}(f) = \cos(2\pi\, fd) + i\sin(2\pi\, fd), \quad f \geq 0 \,. \tag{74}$$

Then by Eq. (59) and twelve applications of [Gradshteyn & Ryzhik, p. 414: 3.741#2] the step response is

$$
\begin{aligned}
r_{\text{Delay}}(t) &= \frac{H_{\text{Delay,Real}}(0)}{2} + \int_0^\infty \frac{H_{\text{Delay,Real}}(f)\sin(2\pi ft) - H_{\text{Delay,Img}}(f)\cos(2\pi ft)}{\pi f}\,df \\
&= \frac{1}{2} + \int_0^\infty \frac{\cos(2\pi fd)\sin(2\pi ft) - \sin(2\pi fd)\cos(2\pi ft)}{\pi f}\,df \\
&= \frac{1}{2} + \begin{cases}
\frac{1}{2} - (0) & \text{if } t > d \text{ and } d \geq 0 \\
\frac{1}{4} - \left(\frac{1}{4}\right) & \text{if } t = d \text{ and } d \geq 0 \\
0 - \left(\frac{1}{2}\right) & \text{if } t < d \text{ and } d \geq 0 \\
\frac{1}{2} - (-0) & \text{if } t > d \text{ and } d < 0 \\
-\frac{1}{4} - \left(-\frac{1}{4}\right) & \text{if } t = d \text{ and } d < 0 \\
-\frac{1}{2} - (0) & \text{if } t < d \text{ and } d < 0
\end{cases} \\
&= \text{step}(t-d),
\end{aligned}
\tag{75}
$$

which of course is a delayed step. It is causal if and only if d is non-negative. See Fig. 8.

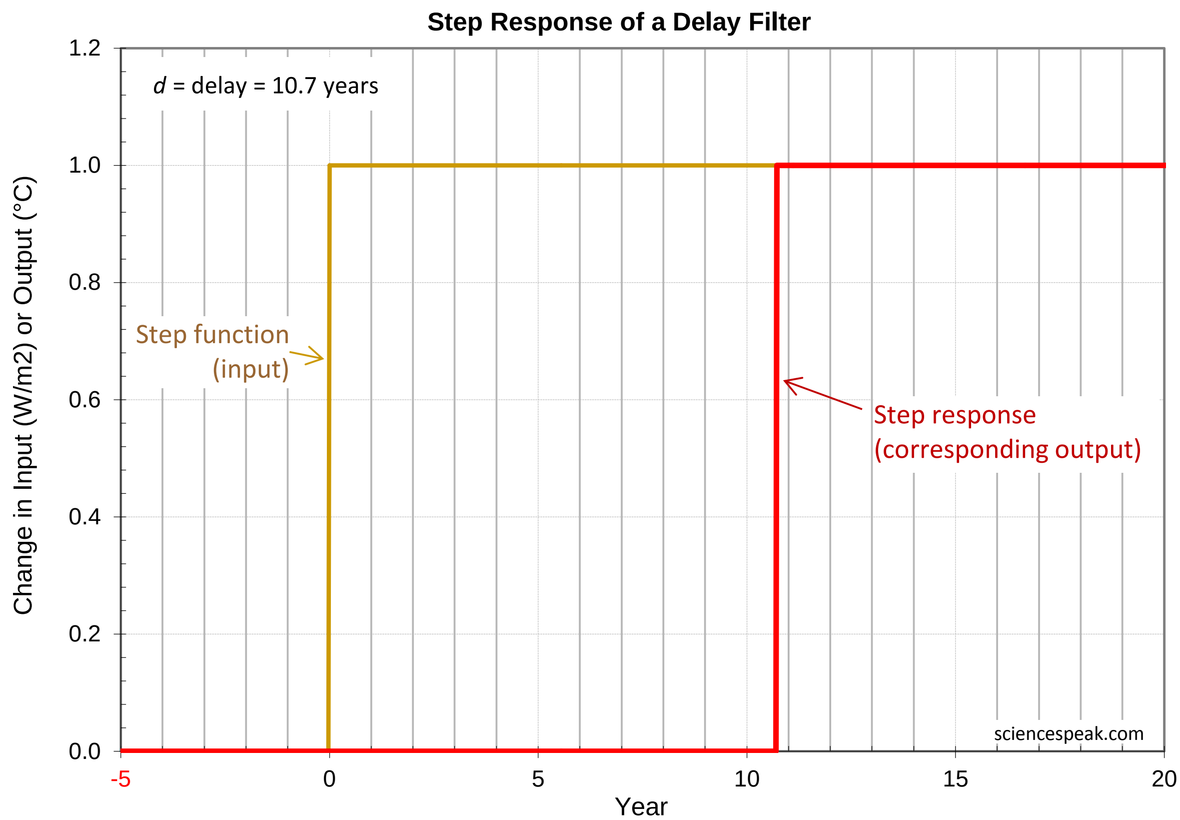


Figure 8: The step response of a delay filter is simply a step function delayed.

The impulse response of the delay filter is, by Eq. (46),

$$\begin{aligned} h_{\text{Delay}}(t) &= \mathbf{B}^{-1}\left\{2^{\eta} H_{\text{Delay}}(f)\right\}(t) \\ &= \int_0^{\infty} 2^{\eta}\left[H_{\text{Delay,real}}(f)\cos(2\pi ft) + H_{\text{Delay,imag}}(f)\sin(2\pi ft)\right] df \\ &= \int_0^{\infty} 2^{\eta}\left[\cos(2\pi fd)\cos(2\pi ft) + \sin(2\pi fd)\sin(2\pi ft)\right] df \\ &= \int_0^{\infty} 2^{\eta} \cos\left[2\pi f(t-d)\right] df \\ &= \int_{-\infty}^{\infty} \cos\left[2\pi f(t-d)\right] df \\ &= \delta(t-d), \end{aligned} \tag{76}$$

where the last equality is justified by symmetry of the integrand in f around zero and that the impulse response of the identity system (when d is zero) is the impulse function.

10 Notch Filters

A notch filter is a system that allows low frequency sinusoids and high frequency sinusoids to pass through with little attenuation, but severely attenuates sinusoids with frequencies around the notch frequency. Also known as a "band-reject" or "band-stop" filter, the amplitude of its transfer function is a near-constant function of frequency, except that near the notch frequency it dips sharply—graphically it looks like a notch. There are many types of notch filters, but here we are only interested in finding the simplest notch filter, its transfer function, and whether or not it is causal.

10.1 The Simplest LIS that is a Notch Filter

10.1.1 A Generic LIS

A generic LIS can be described in the time domain as a linear differential equation whose coefficients are constant (and thus time invariant):

$$\sum_{j=0}^{n} a_j \frac{d^j}{dt^j} g_{\text{OUT}}(t) = \sum_{j=0}^{m} b_j \frac{d^j}{dt^j} g_{\text{IN}}(t), \tag{77}$$

where g_{IN} and g_{OUT} are real-valued input and output functions of time t, m and n are positive integers, and the a_j and b_j coefficients are real constants. Taking the Fourier transform of both sides and applying Eq. (34),

$$\sum_{j=0}^{n} a_j (-i\,2\pi f)^j \mathbf{B}\{g_{\text{OUT}}\}(f) = \sum_{j=0}^{m} b_j (-i\,2\pi f)^j \mathbf{B}\{g_{\text{IN}}\}(f), \quad f \geq 0, \tag{78}$$

so the transfer function of the system is

$$H(f) = \frac{\mathbf{B}\{g_{\text{OUT}}\}(f)}{\mathbf{B}\{g_{\text{IN}}\}(f)} = \frac{\sum_{j=0}^{m} b_j (-i\,2\pi f)^j}{\sum_{j=0}^{n} a_j (-i\,2\pi f)^j}, \qquad f \geq 0. \tag{79}$$

As f becomes much higher or lower than the notch frequency, the only way that $H(f)$ can remain roughly constant is if m equals n. For example, as f increases without limit

$$\lim_{f\to\infty} H(f) \simeq \lim_{f\to\infty} \frac{b_m}{a_n}(-i\,2\pi f)^{m-n}. \tag{80}$$

So the transfer function of a notch filter is the ratio of polynomials of the same degree in f.

10.1.2 A First-Order LIS Is Too Simple to Be a Notch Filter

Is a first-order system ($m = n = 1$) sufficient to build a notch? Eq. (79) becomes

$$H(f) = \frac{b_0 - b_1 i\,2\pi f}{a_0 - a_1 i\,2\pi f} = \frac{b_0}{a_0} \times \frac{1 - i\,f/f_{\mathrm{H}}}{1 - i\,f/f_{\mathrm{L}}}, \qquad f \geq 0, \tag{81}$$

where

$$f_{\mathrm{L}} = \frac{a_0}{2\pi a_1} \qquad \text{and} \qquad f_{\mathrm{H}} = \frac{b_0}{2\pi b_1}$$

are the low and high break frequencies. This transfer function describes a low pass filter with break frequency f_{L} in cascade with a high pass filter with break frequency f_{H}—which means the amplitude of the transfer function on a log-log graph bends down 45° at f_{L} and bends up 45° at f_{H} (Fig. 9). This is not enough bending to create a notch filter (Fig. 9), not even one with a blunt notch, so we need a more complicated LIS.

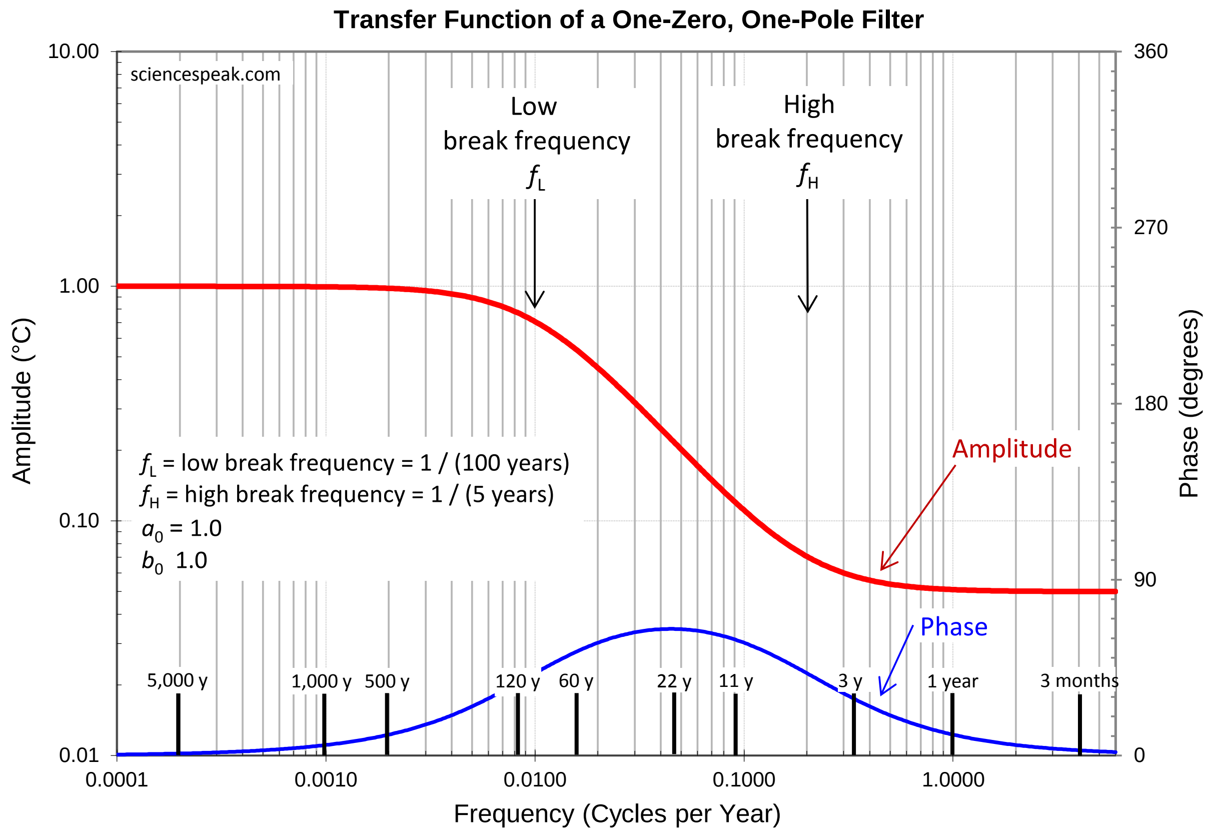


Figure 9: A LIS whose transfer function is a ratio of first-degree polynomials. There are not enough "bends" in the amplitude to be a notch filter—have only two bends.

10.1.3 A Second-Order LIS Can Be a Notch Filter

Is a second-order system ($m = n = 2$) sufficient for a notch? Eq. (79) becomes

$$H(f) = \frac{b_0 + b_1(-i\,2\pi f) + b_2(-i\,2\pi f)^2}{a_0 + a_1(-i\,2\pi f) + a_2(-i\,2\pi f)^2}, \quad f \geq 0. \tag{82}$$

We factorize the quadratic polynomials in the numerator and denominator, treating $\pm i\,2\pi f$ as the polynomial variables (possibly different in the numerator and denominator):

$$H(f) = \frac{\left[(-1)^k i\,2\pi f - z_1\right]\left[(-1)^k i\,2\pi f - z_2\right]}{\left[(-1)^l i\,2\pi f - p_1\right]\left[(-1)^l i\,2\pi f - p_2\right]}, \quad f \geq 0, \tag{83}$$

for some complex numbers z_1, z_2, p_1, p_2, and some binary variables *k* and *l*, called the sign signifiers, that are equal to either 0 or 1 (so there are four different "versions" of this equation—avoiding repetition because most of what we do will be almost identical for all the sign combinations). The a_k and b_j are real, so:

- z_1 and z_2, called the zeroes of the system, are either real or a complex conjugate pair.
- p_1 and p_2, called the poles of the system, are either real or a complex conjugate pair.

The signs of *i* in each of the numerator and denominator of Eq. (83) are treated as separate cases because in the calculations below it is convenient to parameterize the poles and zeroes in polar coordinates using angles that are confined to the first quadrant (the poles and zeroes are confined to the second quadrant of the complex frequency plane). Here we are trying to find all notch-like second-order systems, so we need to consider all possibilities: we can factorize with either sign, independently in both numerator and denominator.

For any given 2nd order transfer function as in Eq. (82), there are in general four transfer functions that differ only by the signs of *i* (fewer than four if either the numerator or denominator are independent of *i*). These four transfer functions all have the same amplitude at each frequency, but their phases differ. As we will show below, two of the transfer functions are for systems with causal step responses, and two have non-causal step responses.

Using the real Fourier transform as defined above, $+i$ in a transfer function is associated with the sinusoid $\sin(2\pi ft)$, so $-i$ represents the sinusoid $-\sin(2\pi ft)$ or $\sin\left[2\pi f(-t)\right]$. Thus the transformation $i \to -i$ in the transfer function changes the sign of the phase change produced by the LIS at each frequency, which in some contexts can be interpreted as reversing the direction of time. This transformation does not affect the magnitude of the transfer function, so it has no effect on whether it is notch-like.

We arrived at Eq. (82) from the original linear differential equation by taking real Fourier transforms and noting that each differentiation in a differential equation is equivalent to multiplying by $-i$ (and also by some real valued function of frequency); see Eq. (34). Although this suggests we have already associated $+i$ with $\sin(2\pi ft)$ because the real FT does, it does not affect the need to consider all possible transfer functions here.

Consider a quadratic q that can be either the numerator or denominator of Eq. (82):

$$q(f) = \left[(-1)^j i\,2\pi f - x_1\right]\left[(-1)^j i\,2\pi f - x_2\right]. \tag{84}$$

If x_1 and x_2 are real numbers then

$$\left|q(f)\right|^2 = \left(4\pi^2 f^2 + x_1^2\right)\left(4\pi^2 f^2 + x_2^2\right) \tag{85}$$

and

$$\frac{d}{df}\left|q(f)\right|^2 = 8\pi^2 f\left(4\pi^2 f^2 + x_2^2\right) + \left(4\pi^2 f^2 + x_1^2\right)8\pi^2 f = 8\pi^2 f\left(8\pi^2 f^2 + x_1^2 + x_2^2\right) \tag{86}$$

which is always greater than zero. Hence there are no maxima or minima in $\left|q(f)\right|$.

If x_1 and x_2 are a complex conjugate pair then $x_1 = a + ib$ and $x_2 = a - ib$ for some real numbers a and b, and

$$q(f) = a^2 + b^2 - 4\pi^2 f^2 - (-1)^j i4\pi af \tag{87}$$

so

$$\left|q(f)\right|^2 = \left(a^2 + b^2 - 4\pi^2 f^2\right)^2 + 16\pi^2 a^2 f^2 \tag{88}$$

and

$$\frac{d}{df}\left|q(f)\right|^2 = 2\left(a^2 + b^2 - 4\pi^2 f^2\right)\left(-8\pi^2 f\right) + 32\pi^2 a^2 f = 16\pi^2 f\left[a^2 - b^2 + 4\pi^2 f^2\right]. \tag{89}$$

Although $\left|q(f)\right|^2$ might have an extreme at $f = 0$, this is not of interest when looking for a notch. More interestingly, $\left|q(f)\right|^2$ has an extreme at

$$f = \frac{\sqrt{b^2 - a^2}}{2\pi} \tag{90}$$

if $|b| > |a|$, and the second derivative at this extreme is

$$\left.\frac{d^2}{df^2}\left|q(f)\right|^2\right|_{f=\frac{\sqrt{b^2-a^2}}{2\pi}} = 16\pi^2\left[a^2 - b^2 + 12\pi^2 f^2\right]_{f=\frac{\sqrt{b^2-a^2}}{2\pi}} = 32\pi^2(b^2 - a^2), \tag{91}$$

which is negative—so the extreme is a minimum.

Now we can construct a notch filter from Eq. (82). Given that its numerator and denominator are each either flat or have a minimum, a minimum in $\left|H(f)\right|$ can only arise at a frequency where the numerator has a minimum and the denominator is varying slowly. Therefore:

- The zeroes z_1 and z_2 are a complex conjugate pair $a \pm ib$ where $|b| > |a|$, and the minimum occurs near the frequency f in Eq. (90).
- The poles p_1 and p_2 are either complex conjugates (somewhere near the zeroes, to avoid creating a maximum far from the minimum which would detract from the notch-like shape) or real numbers. Unfortunately the transfer functions for the com-

plex and real cases are structurally different, so they have to be treated separately in the analysis below.

This demonstrates that a system whose transfer function is a ratio of quadratics can be a notch filter, and that it is the lowest order system that can be a notch filter.

10.1.4 Higher-Order Notch Filters

The general form of a filter is the transfer function of Eq. (79), a ratio of polynomials in f with real coefficients. Any sufficiently continuous function is sufficiently closely approximated by polynomials (at least piecewise), so transfer functions of physical interest, certainly those corresponding to linear differential equations, can be expressed as such a filter.

By the fundamental theorem of algebra, the polynomials in Eq. (79) are factorable into the product of irreducible first and second degree polynomials, the former having one real root and the latter two complex-conjugate roots. Any LIS or filter can only be constructed as the product of such first and second order notch filters, so clearly a higher order notch filter must be the product of second order notch filters that share the same notch frequency. Hence LISs that are notch filters are cascades of second order notch filters.

10.2 Second-Order Notch Filters

Engineers design filters in the complex-frequency plane. We are not going to explain that methodology here, except to motivate some remarks about parameterization by showing the poles and zeroes of a second-order filter (Eq. (82)) in the complex frequency plane in Fig. 10. (We won't use these notations here, but beware if looking in the electrical engineering literature: electrical engineers use "i" for current, "j" for the square root of -1, and "ω" for frequency measured in radians per unit time, i.e. $\omega = 2\pi f$.)

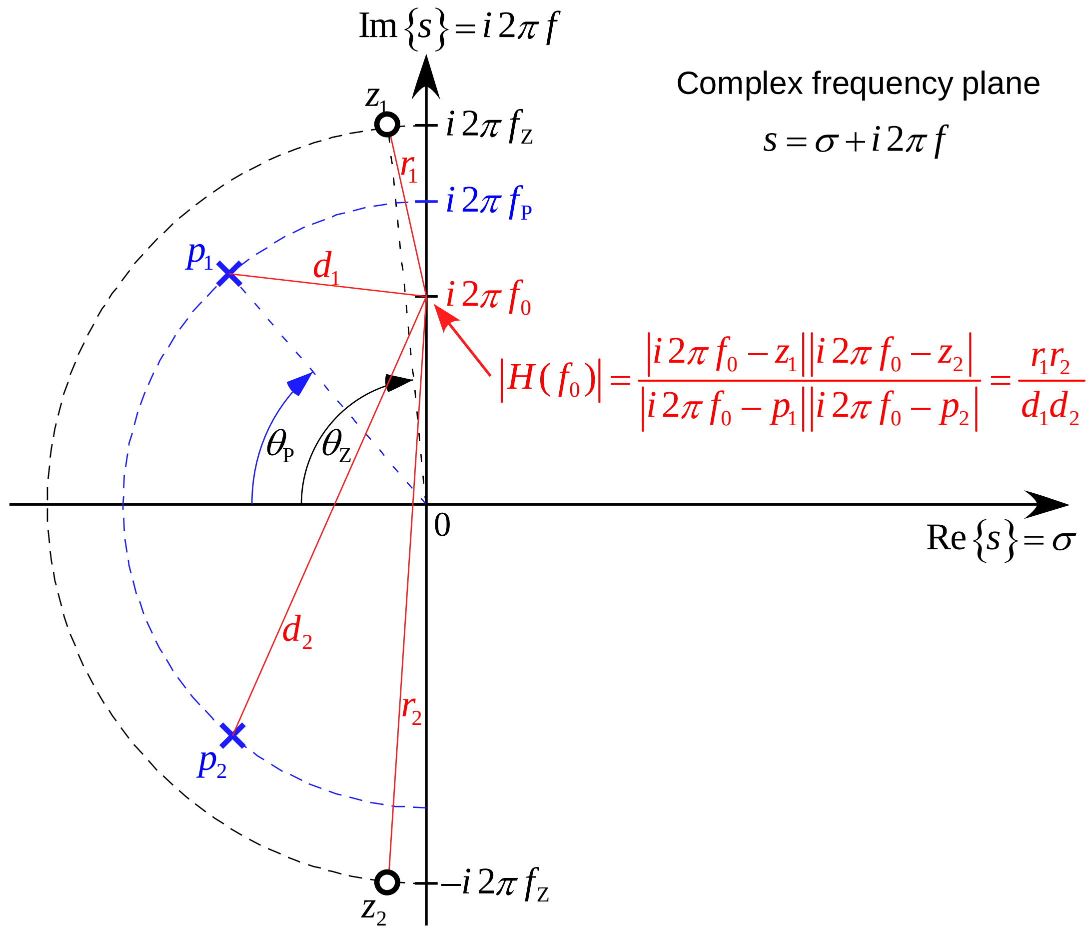


Figure 10: The poles and zeroes of a second order filter, in the complex frequency plane.

The complex frequency variable s is shown in Fig. 10, on a complex plane. The imaginary axis is vertical, and shows (real) frequencies. The real axis is horizontal, and shows rates of exponential contraction (left of zero) or expansion (right side). The zeroes and poles of Eq. (82) are marked as black zeroes and blue crosses respectively; they are each complex conjugate pairs. Our transfer function H is only defined on the imaginary axis. (Engineers often define the transfer function for any point on the plane; they then call the function whose domain is confined to the frequency axis "the frequency response".)

The amplitude of H at some frequency f_0 is shown in red, evaluated as the ratio of products of distances from f_0 to the poles and zeroes. The product $r_1 r_2$ is minimized when f_0 is near the zero frequency f_Z and is otherwise fairly flat, which is in accord with our calculation in involving Eq. (89). But $1/d_1 d_2$ is maximized when f_0 is near f_P, so f_P should be near f_Z and the poles should be to the left of the zeroes in Fig. 10 to ensure the minimization due to the zeroes is greater than the maximization due to the poles. Engineering experience and theory further informs us that:

- For a sharp notch, the zeroes should be close to the imaginary axis.
- Putting the zeroes and poles the same distance from the origin ensures that $|H(f)|$ is near unity except near the notch frequency.
- For stability of the system, the poles *must* be in the left half plane, i.e. have a negative real part. (The transfer function, and thus the amplification of the system, is infinite at

a complex frequency where there is a pole. If such a pole is in the right half plane, where the exponential parts of the complex frequencies are positive, the pole would amplify an exponentially-growing sinusoid infinitely—the system output would grow without limit, causing instability.) Keeping the zeroes in the left hand plane might be wise for stability.

10.2.1 Complex Poles

Fig. 10 inspires the following parameterizations of the transfer function in Eq. (83), using polar coordinates. The zeroes z_1 and z_2 are a complex conjugate pair, while the poles p_1 and p_2 are also a complex conjugate pair, so let

$$\begin{aligned} z_1, z_2 &= 2\pi f_Z \exp\left[\pm i(\pi - \theta_Z)\right] = 2\pi f_Z \left(-\cos\theta_Z \pm i \sin\theta_Z\right) \\ p_1, p_2 &= 2\pi f_P \exp\left[\pm i(\pi - \theta_P)\right] = 2\pi f_P \left(-\cos\theta_P \pm i \sin\theta_P\right), \end{aligned} \tag{92}$$

where the zero and pole frequencies f_Z and f_P (the reciprocals of the zero and pole periods) and the zero and pole angles θ_Z and θ_P are as shown in Fig. 10. Then the numerator of the transfer function in Eq. (83) is

$$\begin{aligned} &\left[(-1)^k i\, 2\pi f - z_1\right]\left[(-1)^k i\, 2\pi f - z_2\right] \\ &= \left[(-1)^k i\, 2\pi f - 2\pi f_Z\left(-\cos\theta_Z + i \sin\theta_Z\right)\right]\left[(-1)^k i\, 2\pi f - 2\pi f_Z\left(-\cos\theta_Z - i \sin\theta_Z\right)\right] \\ &= 4\pi^2\left[(-1)^k i f + f_Z \cos\theta_Z - i f_Z \sin\theta_Z\right]\left[(-1)^k i f + f_Z \cos\theta_Z + i f_Z \sin\theta_Z\right] \\ &= 4\pi^2\left\{\left[(-1)^k i f + f_Z \cos\theta_Z\right]^2 - \left(i f_Z \sin\theta_Z\right)^2\right\} \\ &= 4\pi^2\left[-f^2 + (-1)^k i\, 2 f f_Z \cos\theta_Z + f_Z^2 \cos^2\theta_Z + f_Z^2 \sin^2\theta_Z\right] \\ &= 4\pi^2\left[-f^2 + (-1)^k i\, 2 f f_Z \cos\theta_Z + f_Z^2\right]. \end{aligned} \tag{93}$$

Similarly for the denominator. Thus the transfer function of a second-order notch filter with complex poles is

$$H_{\mathrm{Notch},\mathbb{C}}(f) = \frac{f_Z^2 - f^2 + (-1)^k i\, 2 f f_Z \cos\theta_Z}{f_P^2 - f^2 + (-1)^l i\, 2 f f_P \cos\theta_P}, \qquad f \geq 0, \tag{94}$$

where the four real parameters are constrained as

$$f_Z > 0,\ \theta_Z \in \left(45°, 90°\right),\ f_P > 0,\ \theta_P \in \left(0, 90°\right). \tag{95}$$

Note that not all parameters that meet these constraints will make H_{Notch} a notch filter (generally speaking, that requires θ_Z near 90°, $\theta_P < \theta_Z$, and f_P vaguely near f_Z). For sharper cutoffs or deeper notches (as required in electronics), use a higher order filter—which are just products of such second order notch filters. See Fig. 11.

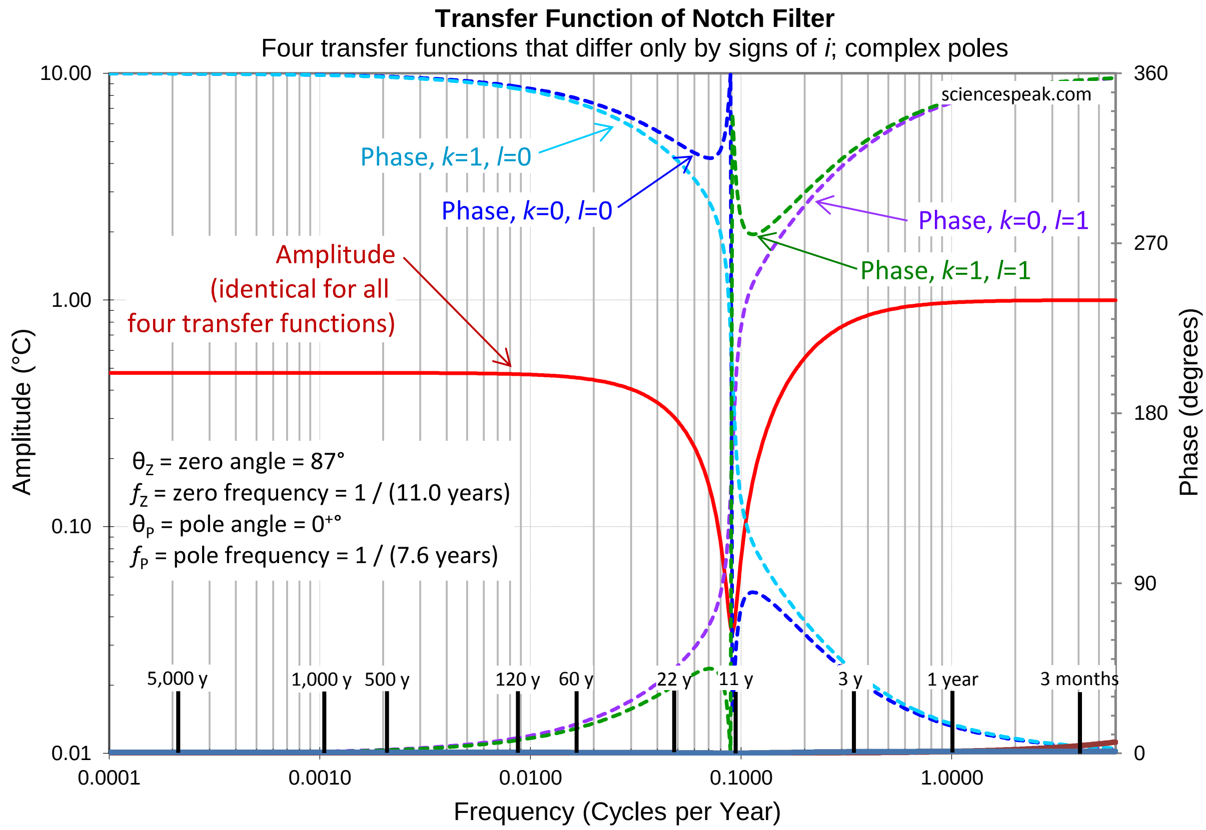


Figure 11: The four notch filters with a given combination of two zeroes and two complex poles as parameterized here. The four transfer functions differ only in signs of *i* (the binary variables *k* and *l*), so their amplitudes are identical and only their phases differ. Note the "notch"—the sharp reduction in amplitude around 11 years—and the uneven "shoulders". While not the sharpest, this type of notch filter is the simplest.

10.2.2 Real Poles

As in the complex case, the zeroes z_1 and z_2 are a complex conjugate pair, and we use the same parameterization. However the poles p_1 and p_2 are real. For system stability they must both be in the left half of the complex frequency plane in Fig. 10, so they are negative:

$$p_1 < 0, \quad p_2 < 0. \tag{96}$$

The complex- and real-pole cases coincide when the poles are the same, both on the real axis at a distance D to the left of the origin in Fig. 10:

- In the complex case, θ_P is zero and D is $2\pi f_P$.
- In the real case, p_1 and p_2 are both equal to $-D$.

To make the algebra below easier, and to parallel the frequencies f_P and f_Z in the complex case, we define the radial exponential decay constants d_1 and d_2 by

$$p_1 = -2\pi d_1, \quad p_2 = -2\pi d_2. \tag{97}$$

Thus $2\pi d_1$ and $2\pi d_2$ are the distances from the origin leftwards to the poles in Fig. 10. They are the multipliers of time in an exponential decay term, their units are inverse-time, and their inverses are exponential decay constants.

The denominator of the transfer function in Eq. (83) is then

$$\begin{aligned}\left[(-1)^l i\,2\pi f - p_1\right]\left[(-1)^l i\,2\pi f - p_2\right] &= \left[(-1)^l i\,2\pi f + 2\pi d_1\right]\left[(-1)^l i\,2\pi f + 2\pi d_2\right] \\ &= 4\pi^2\left[(-1)^l i\, f + d_1\right]\left[(-1)^l i\, f + d_2\right] \\ &= 4\pi^2\left[d_1 d_2 - f^2 + (-1)^l i\, f\left(d_1 + d_2\right)\right],\end{aligned} \tag{98}$$

so, by Eq. (93), the transfer function of a second order notch filter with real poles is

$$H_{\text{Notch},\mathbb{R}}(f) = \frac{f_Z^2 - f^2 + (-1)^k i\,2f\,f_Z\cos\theta_Z}{d_1 d_2 - f^2 + (-1)^l i\, f\left(d_1 + d_2\right)}, \qquad f \ge 0, \tag{99}$$

where the four real parameters are constrained as

$$f_Z > 0,\ \theta_Z \in \left(45°, 90°\right),\ d_1 > 0,\ d_2 > 0,\ d_1 \ne d_2. \tag{100}$$

There is little difference in the magnitude of the transfer function between real and complex poles; a given transfer function can apparently be constructed from either. See Fig. 12.

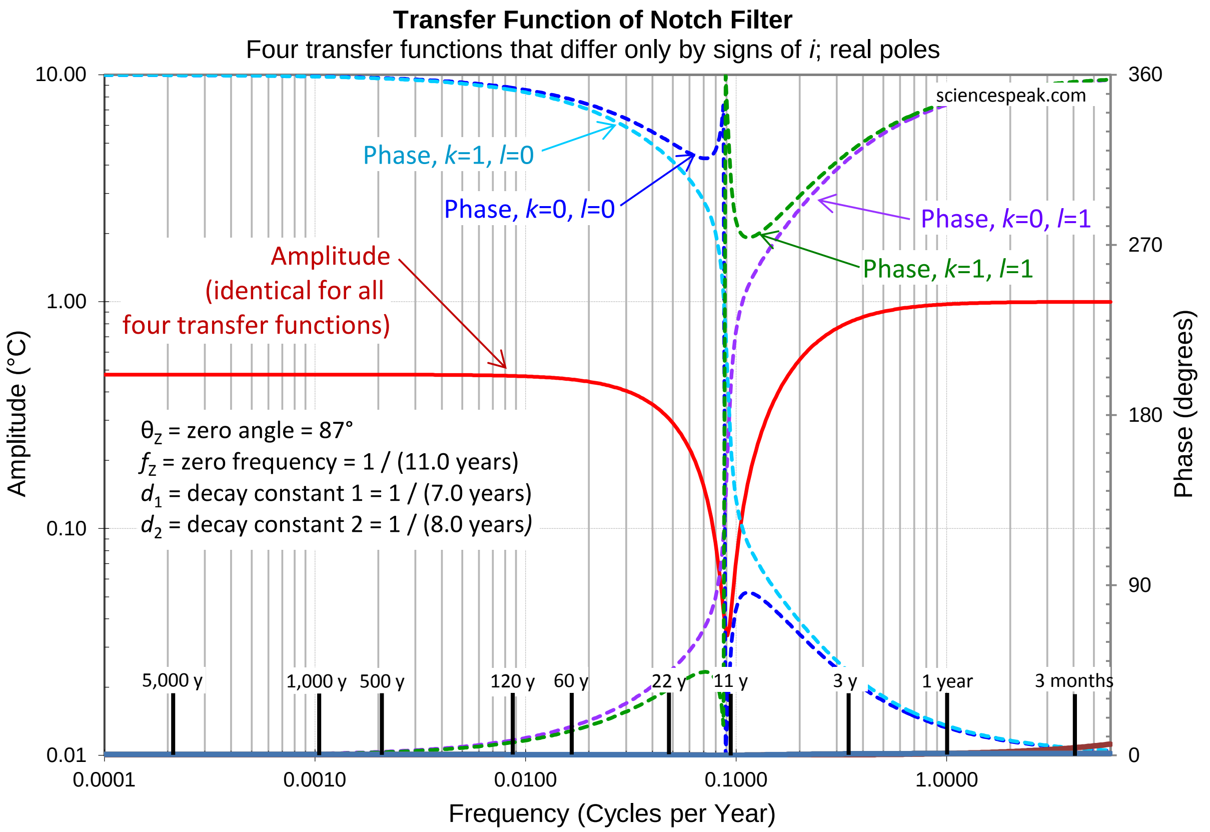


Figure 12: The four notch filters with a given combination of two zeroes and two real poles as parameterized here. Almost indistinguishable from Fig. 11 (though they *are* different).

10.3 Step Response of a Second-Order Notch Filter

10.3.1 Complex Poles

First split the transfer function of the notch filter in Eq. (94) into real and imaginary parts:

$$
\begin{aligned}
H_{\text{Notch},\mathbb{C}}(f) &= \frac{\left[f_{\text{Z}}^2 - f^2 + (-1)^k i 2 f f_{\text{Z}} \cos\theta_{\text{Z}} \right]\left[f_{\text{P}}^2 - f^2 - (-1)^l i 2 f f_{\text{P}} \cos\theta_{\text{P}} \right]}{\left(f_{\text{P}}^2 - f^2 \right)^2 + 4 f^2 f_{\text{P}}^2 \cos^2\theta_{\text{P}}} \\
&= \frac{\left\{ \begin{aligned} &\left[\left(f_{\text{Z}}^2 - f^2 \right)\left(f_{\text{P}}^2 - f^2 \right) + (-1)^{k+l} 4 f^2 f_{\text{Z}} f_{\text{P}} \cos\theta_{\text{Z}} \cos\theta_{\text{P}} \right] \\ &+ i 2 f \left[\left(f_{\text{P}}^2 - f^2 \right)(-1)^k f_{\text{Z}} \cos\theta_{\text{Z}} - \left(f_{\text{Z}}^2 - f^2 \right)(-1)^l f_{\text{P}} \cos\theta_{\text{P}} \right] \end{aligned} \right\}}{f_{\text{P}}^4 + f^4 + 2 f_{\text{P}}^2 f^2 \left(2\cos^2\theta_{\text{P}} - 1 \right)} \\
&= \frac{\left(f^4 + \alpha f^2 + \beta \right) + i\left(\gamma f^3 + \mu f \right)}{f^4 + 2 f_{\text{P}}^2 f^2 \cos\left(2\theta_{\text{P}} \right) + f_{\text{P}}^4}, \quad f \geq 0,
\end{aligned} \tag{101}
$$

where

$$
\begin{aligned}
\alpha &= -f_{\text{Z}}^2 - f_{\text{P}}^2 + \left[(-1)^k 2 f_{\text{Z}} \cos\theta_{\text{Z}} \right]\left[(-1)^l 2 f_{\text{P}} \cos\theta_{\text{P}} \right] \\
\beta &= f_{\text{Z}}^2 f_{\text{P}}^2 \\
\gamma &= \left[(-1)^l 2 f_{\text{P}} \cos\theta_{\text{P}} \right] - \left[(-1)^k 2 f_{\text{Z}} \cos\theta_{\text{Z}} \right] \\
\mu &= f_{\text{P}}^2 \left[(-1)^k 2 f_{\text{Z}} \cos\theta_{\text{Z}} \right] - f_{\text{Z}}^2 \left[(-1)^l 2 f_{\text{P}} \cos\theta_{\text{P}} \right].
\end{aligned} \tag{102}
$$

Then by Eq. (59) the step response of the notch filter is

$$
\begin{aligned}
r_{\text{Notch},\mathbb{C}}(t) &= \frac{H_{\text{Notch,Real}}(0)}{2} + \int_0^\infty \frac{H_{\text{Notch,Real}}(f)\sin\left(2\pi f t \right) - H_{\text{Notch,Img}}(f)\cos\left(2\pi f t \right)}{\pi f} df \\
&= \frac{\beta}{2 f_{\text{P}}^4} + \int_0^\infty \frac{f^4 + \alpha f^2 + \beta}{\pi f \left[f^4 + 2 f_{\text{P}}^2 f^2 \cos\left(2\theta_{\text{P}} \right) + f_{\text{P}}^4 \right]} \sin\left(2\pi f t \right) df \\
&\quad - \int_0^\infty \frac{\gamma f^2 + \mu}{\pi \left[f^4 + 2 f_{\text{P}}^2 f^2 \cos\left(2\theta_{\text{P}} \right) + f_{\text{P}}^4 \right]} \cos\left(2\pi f t \right) df .
\end{aligned} \tag{103}
$$

This integration was performed numerically (in polar form, Eq. (63)), and gives the same result as the following calculation. By [Gradshteyn & Ryzhik, 1980, pp. 411, 3.733#4,#2,#5],

$$
\begin{aligned}
&\int_0^\infty \frac{f^4 + \alpha f^2 + \beta}{\pi f \left[f^4 + 2 f_{\text{P}}^2 f^2 \cos\left(2\theta_{\text{P}} \right) + f_{\text{P}}^4 \right]} \sin\left(2\pi f t \right) df \\
&\quad = \frac{1}{2} \exp\left(-2\pi |t| f_{\text{P}} \cos\theta_{\text{P}} \right) \frac{\sin\left(2\theta_{\text{P}} - 2\pi f_{\text{P}} |t| \sin\theta_{\text{P}} \right)}{\sin\left(2\theta_{\text{P}} \right)} \operatorname{sgn}(t) \\
&\quad + \frac{\alpha}{2 f_{\text{P}}^2} \exp\left(-2\pi |t| f_{\text{P}} \cos\theta_{\text{P}} \right) \frac{\sin\left(2\pi f_{\text{P}} |t| \sin\theta_{\text{P}} \right)}{\sin\left(2\theta_{\text{P}} \right)} \operatorname{sgn}(t) \\
&\quad + \frac{\beta}{2 f_{\text{P}}^4} \left[1 - \exp\left(-2\pi |t| f_{\text{P}} \cos\theta_{\text{P}} \right) \frac{\sin\left(2\theta_{\text{P}} + 2\pi f_{\text{P}} |t| \sin\theta_{\text{P}} \right)}{\sin\left(2\theta_{\text{P}} \right)} \right] \operatorname{sgn}(t).
\end{aligned} \tag{104}
$$

By [Gradshteyn & Ryzhik, 1980, pp. 411, 3.733#3,#1],

$$-\int_0^\infty \frac{\gamma f^2+\lambda}{\pi\left[f^4+2f_{\mathrm{P}}^2 f^2\cos\left(2\theta_{\mathrm{P}}\right)+f_{\mathrm{P}}^4\right]}\cos\left(2\pi ft\right)df$$

$$=-\frac{\gamma}{2f_{\mathrm{P}}}\exp\left(-2\pi f_{\mathrm{P}}|t|\cos\theta_{\mathrm{P}}\right)\frac{\sin\left(\theta_{\mathrm{P}}-2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right)}{\sin\left(2\theta_{\mathrm{P}}\right)} \quad (105)$$

$$-\frac{\mu}{2f_{\mathrm{P}}^3}\exp\left(-2\pi f_{\mathrm{P}}|t|\cos\theta_{\mathrm{P}}\right)\frac{\sin\left(\theta_{\mathrm{P}}+2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right)}{\sin\left(2\theta_{\mathrm{P}}\right)}.$$

Thus

$$r_{\mathrm{Notch},\mathbb{C}}(t)=\frac{\beta}{2f_{\mathrm{P}}^4}\left[1+\mathrm{sgn}(t)\right]+\frac{\exp\left(-2\pi f_{\mathrm{P}}|t|\cos\theta_{\mathrm{P}}\right)}{2\sin\left(2\theta_{\mathrm{P}}\right)}\Lambda(t), \quad (106)$$

where

$$\begin{aligned}\Lambda(t)= &\ \sin\left(2\theta_{\mathrm{P}}-2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right)\mathrm{sgn}(t)\\ &+\alpha f_{\mathrm{P}}^{-2}\sin\left(2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right)\mathrm{sgn}(t)\\ &-\beta f_{\mathrm{P}}^{-4}\sin\left(2\theta_{\mathrm{P}}+2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right)\mathrm{sgn}(t)\\ &-\gamma f_{\mathrm{P}}^{-1}\sin\left(\theta_{\mathrm{P}}-2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right)\\ &-\mu f_{\mathrm{P}}^{-3}\sin\left(\theta_{\mathrm{P}}+2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}\right).\end{aligned} \quad (107)$$

Let

$$\rho=f_{\mathrm{Z}}/f_{\mathrm{P}} \quad \text{and} \quad \xi=2\pi|t|f_{\mathrm{P}}\sin\theta_{\mathrm{P}}. \quad (108)$$

Then by Eq. (102)

$$r_{\mathrm{Notch},\mathbb{C}}(t)=\rho^2\,\mathrm{step}(t)+\frac{\exp\left(-2\pi f_{\mathrm{P}}|t|\cos\theta_{\mathrm{P}}\right)}{4\sin\theta_{\mathrm{P}}\cos\theta_{\mathrm{P}}}\Lambda(t) \quad (109)$$

where

$$\begin{aligned}\Lambda(t)= &\ \sin\left(2\theta_{\mathrm{P}}-\xi\right)\mathrm{sgn}(t)\\ &+\left[-\rho^2-1+(-1)^{k+l}4\rho\cos\theta_{\mathrm{Z}}\cos\theta_{\mathrm{P}}\right]\sin\left(\xi\right)\mathrm{sgn}(t)\\ &-\rho^2\sin\left(2\theta_{\mathrm{P}}+\xi\right)\mathrm{sgn}(t)\\ &-\left[(-1)^l 2\cos\theta_{\mathrm{P}}-(-1)^k 2\rho\cos\theta_{\mathrm{Z}}\right]\sin\left(\theta_{\mathrm{P}}-\xi\right)\\ &-\left[(-1)^k 2\rho\cos\theta_{\mathrm{Z}}-(-1)^l 2\rho^2\cos\theta_{\mathrm{P}}\right]\sin\left(\theta_{\mathrm{P}}+\xi\right).\end{aligned} \quad (110)$$

Expanding the trigonometric functions in Λ:

$$
\begin{aligned}
\Lambda(t) = &\left[2\sin\theta_\mathrm{P}\cos\theta_\mathrm{P}\cos\xi - \cos^2\theta_\mathrm{P}\sin\xi + \sin^2\theta_\mathrm{P}\sin\xi\right]\mathrm{sgn}(t) \\
&+\left(-\rho^2 - 1 + (-1)^{k+l}4\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P}\right)\left[\sin\xi\right]\mathrm{sgn}(t) \\
&-\rho^2\left[2\sin\theta_\mathrm{P}\cos\theta_\mathrm{P}\cos\xi + \cos^2\theta_\mathrm{P}\sin\xi - \sin^2\theta_\mathrm{P}\sin\xi\right]\mathrm{sgn}(t) \\
&-\left[(-1)^l 2\cos\theta_\mathrm{P} - (-1)^k 2\rho\cos\theta_\mathrm{Z}\right]\left[\sin\theta_\mathrm{P}\cos\xi - \cos\theta_\mathrm{P}\sin\xi\right] \\
&-\left[(-1)^k 2\rho\cos\theta_\mathrm{Z} - (-1)^l 2\rho^2\cos\theta_\mathrm{P}\right]\left[\sin\theta_\mathrm{P}\cos\xi + \cos\theta_\mathrm{P}\sin\xi\right]
\end{aligned}
\tag{111}
$$

Collecting terms:

$$
\begin{aligned}
\Lambda(t) = &\begin{bmatrix}
+2\sin\theta_\mathrm{P}\cos\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
-2\rho^2\sin\theta_\mathrm{P}\cos\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
-(-1)^l 2\cos\theta_\mathrm{P}\sin\theta_\mathrm{P} + (-1)^k 2\rho\cos\theta_\mathrm{Z}\sin\theta_\mathrm{P} \\
-(-1)^k 2\rho\cos\theta_\mathrm{Z}\sin\theta_\mathrm{P} + (-1)^l 2\rho^2\cos\theta_\mathrm{P}\sin\theta_\mathrm{P}
\end{bmatrix}\cos\xi \\
&+\begin{bmatrix}
-\cos^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+\sin^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+\left(-\rho^2 - 1 + (-1)^{k+l}4\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P}\right)\mathrm{sgn}(t) \\
-\rho^2\cos^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+\rho^2\sin^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+(-1)^l 2\cos^2\theta_\mathrm{P} - (-1)^k 2\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P} \\
-(-1)^k 2\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P} + (-1)^l 2\rho^2\cos^2\theta_\mathrm{P}
\end{bmatrix}\sin\xi .
\end{aligned}
\tag{112}
$$

Simplifying:

$$
\begin{aligned}
\Lambda(t) = &\begin{bmatrix}
+2\left(1-\rho^2\right)\cos\theta_\mathrm{P}\sin\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
-(-1)^l 2\left(1-\rho^2\right)\cos\theta_\mathrm{P}\sin\theta_\mathrm{P}
\end{bmatrix}\cos\xi \\
&+\begin{bmatrix}
-\left(1+\rho^2\right)\cos^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+\left(1+\rho^2\right)\sin^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+\left(-\rho^2 - 1 + (-1)^{k+l}4\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P}\right)\mathrm{sgn}(t) \\
+(-1)^l 2\left(1+\rho^2\right)\cos^2\theta_\mathrm{P} \\
-(-1)^k 4\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P}
\end{bmatrix}\sin\xi \\
= &\ 2\left(1-\rho^2\right)\cos\theta_\mathrm{P}\sin\theta_\mathrm{P}\left[\mathrm{sgn}(t) - (-1)^l\right]\cos\xi \\
&+2\begin{bmatrix}
-\left(1+\rho^2\right)\cos^2\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
+(-1)^{k+l}2\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P}\,\mathrm{sgn}(t) \\
(-1)^l\left(1+\rho^2\right)\cos^2\theta_\mathrm{P} \\
-(-1)^k 2\rho\cos\theta_\mathrm{Z}\cos\theta_\mathrm{P}
\end{bmatrix}\sin\xi
\end{aligned}
\tag{113}
$$

so

$$\begin{aligned}
\Lambda(t) &= 2\left(1-\rho^2\right)\cos\theta_{\mathrm{P}}\sin\theta_{\mathrm{P}}\left[\operatorname{sgn}(t)-(-1)^l\right]\cos\xi \\
&\quad +2\left\{\begin{aligned}&\left(1+\rho^2\right)\cos^2\theta_{\mathrm{P}}\left[(-1)^l-\operatorname{sgn}(t)\right] \\ &+2\rho\cos\theta_{\mathrm{Z}}\cos\theta_{\mathrm{P}}\left[(-1)^{k+l}\operatorname{sgn}(t)-(-1)^k\right]\end{aligned}\right\}\sin\xi. \\
&= 2\left(1-\rho^2\right)\cos\theta_{\mathrm{P}}\sin\theta_{\mathrm{P}}\left[\operatorname{sgn}(t)-(-1)^l\right]\cos\xi \\
&\quad +2\left[-\left(1+\rho^2\right)\cos^2\theta_{\mathrm{P}}+2\rho\cos\theta_{\mathrm{Z}}\cos\theta_{\mathrm{P}}(-1)^{k+l}\right]\left[\operatorname{sgn}(t)-(-1)^l\right]\sin\xi
\end{aligned} \tag{114}$$

and finally

$$\Lambda(t) = 2\left[\operatorname{sgn}(t)-(-1)^l\right]\left\{\begin{aligned}&\left(1-\rho^2\right)\cos\theta_{\mathrm{P}}\sin\theta_{\mathrm{P}}\cos\xi \\ &+\left[\begin{aligned}&(-1)^{k+l}2\rho\cos\theta_{\mathrm{Z}}\cos\theta_{\mathrm{P}} \\ &-\left(1+\rho^2\right)\cos^2\theta_{\mathrm{P}}\end{aligned}\right]\sin\xi\end{aligned}\right\}. \tag{115}$$

Thus the step response of the 2nd order notch filter with complex poles, whose transfer function is in Eq. (94), is (finally!)

$$r_{\mathrm{Notch},\mathbb{C}}(t) = \rho^2\operatorname{step}(t)+\frac{1}{2}\left[\operatorname{sgn}(t)-(-1)^l\right]\exp\left(-\omega_{\mathrm{C}}|t|\right)\left[A_{\mathbb{C}}\cos\left(\omega_{\mathrm{S}}t\right)+B_{\mathbb{C}}\sin\left(\omega_{\mathrm{S}}|t|\right)\right] \tag{116}$$

where the constants (independent of *t*) are

$$\begin{aligned}
A_{\mathbb{C}} &= 1-\rho^2 & B_{\mathbb{C}} &= \frac{(-1)^{k+l}2\rho\cos\theta_{\mathrm{Z}}-\left(1+\rho^2\right)\cos\theta_{\mathrm{P}}}{\sin\theta_{\mathrm{P}}} \\
\omega_{\mathrm{C}} &= 2\pi f_{\mathrm{P}}\cos\theta_{\mathrm{P}} & \rho &= f_{\mathrm{Z}}/f_{\mathrm{P}} \\
\omega_{\mathrm{S}} &= 2\pi f_{\mathrm{P}}\sin\theta_{\mathrm{P}}.
\end{aligned} \tag{117}$$

Note that $\omega_{\mathrm{C}} > 0$, $\omega_{\mathrm{S}} > 0$, and $\rho > 0$.

The character of the step response depends decisively on *l*. For complex poles, if *l* is zero then the step response is

$$r_{\mathrm{Notch},\mathbb{C},0}(t) = \begin{cases} -\exp\left(\omega_{\mathrm{C}}t\right)\left[A_{\mathbb{C}}\cos\left(\omega_{\mathrm{S}}t\right)-B_{\mathbb{C}}\sin\left(\omega_{\mathrm{S}}t\right)\right] & \text{if } t<0 \\ \rho^2-\dfrac{1}{2} & \text{if } t=0 \\ \rho^2 & \text{if } t>0, \end{cases} \tag{118}$$

which is non-causal (that is, non-zero before the step stimulus starts at time zero). However if *l* is one then the step response is causal:

$$r_{\mathrm{Notch},\mathbb{C},1}(t) = \begin{cases} 0 & \text{if } t<0 \\ \dfrac{1}{2} & \text{if } t=0 \\ \rho^2+\exp\left(-\omega_{\mathrm{C}}t\right)\left[A_{\mathbb{C}}\cos\left(\omega_{\mathrm{S}}t\right)+B_{\mathbb{C}}\sin\left(\omega_{\mathrm{S}}t\right)\right] & \text{if } t>0. \end{cases} \tag{119}$$

The effect of k is confined to $B_{\mathbb{C}}$, which does not affect the character of the step response. l also affects $B_{\mathbb{C}}$.

This result has been checked against numerical integrations of Eq. (101) in Eq. (63). An example is shown in Fig. 13.

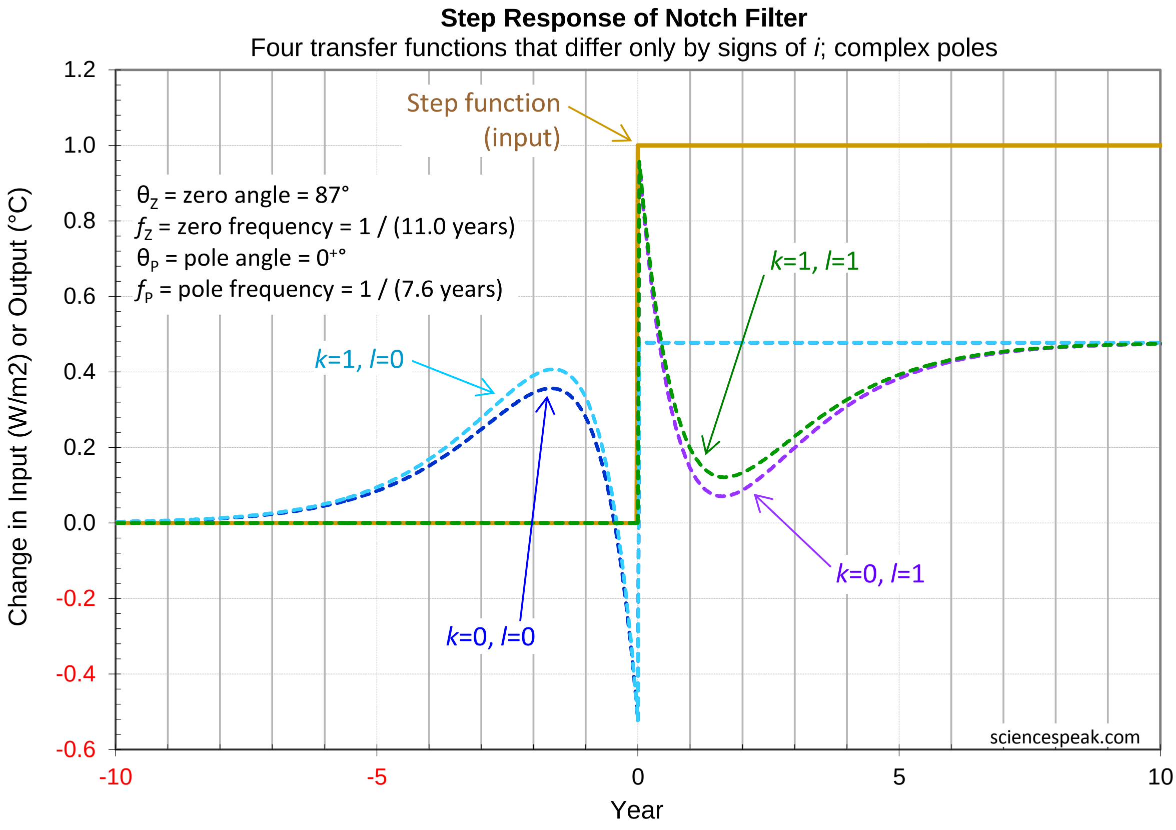


Figure 13: Step response of the four notch filters in Fig. 11, with complex poles. The four notch filters, sharing the same combination of two zeroes and two complex poles as parameterized here, have transfer functions that differ only in signs of *i* (the binary variables *k* and *l*). When *l* is 1 the step response is casual, but when *l* is 0 the step response is non-causal (i.e. the step response starts before the stimulus, that is, before the step function rises).

10.3.2 Real Poles

First split the transfer function of the notch filter in Eq. (99) into real and imaginary parts:

$$H_{\text{Notch},\mathbb{R}}(f) = \frac{\left(f_Z^2 - f^2 + (-1)^k i\, 2 f f_Z \cos\theta_Z\right)\left[d_1 d_2 - f^2 - (-1)^l i f (d_1 + d_2)\right]}{\left(d_1 d_2 - f^2\right)^2 + f^2 \left(d_1 + d_2\right)^2}$$

$$= \frac{\left\{\begin{array}{l} \left[\left(f_Z^2 - f^2\right)\left(d_1 d_2 - f^2\right) + (-1)^{k+l} 2 f^2 (d_1 + d_2) f_Z \cos\theta_Z\right] \\ + i f\left[\left(d_1 d_2 - f^2\right)(-1)^k 2 f_Z \cos\theta_Z - \left(f_Z^2 - f^2\right)(-1)^l (d_1 + d_2)\right] \end{array}\right\}}{d_1^2 d_2^2 + f^4 + f^2\left(d_1^2 + d_2^2\right)}$$

$$= \frac{\left(f^4 + \varepsilon f^2 + \zeta\right) + i\left(\nu f^3 + \tau f\right)}{\left(f^2 + d_1^2\right)\left(f^2 + d_2^2\right)}, \quad f \geq 0, \tag{120}$$

where

$$\begin{aligned}
\varepsilon &= -f_Z^2 - d_1 d_2 + \left[(-1)^k 2 f_Z \cos\theta_Z\right]\left[(-1)^l \left(d_1 + d_2\right)\right] \\
\zeta &= f_Z^2 d_1 d_2 \\
\nu &= \left[(-1)^l \left(d_1 + d_2\right)\right] - \left[(-1)^k 2 f_Z \cos\theta_Z\right] \\
\tau &= d_1 d_2 \left[(-1)^k 2 f_Z \cos\theta_Z\right] - f_Z^2 \left[(-1)^l \left(d_1 + d_2\right)\right].
\end{aligned} \tag{121}$$

Then by Eq. (59) the step response of the notch filter is

$$\begin{aligned}
r_{\text{Notch},\mathbb{R}}(t) &= \frac{H_{\text{Notch},\mathbb{R},\text{Real}}(0)}{2} + \int_0^\infty \frac{H_{\text{Notch},\mathbb{R},\text{Real}}(f)\sin\left(2\pi ft\right) - H_{\text{Notch},\mathbb{R},\text{Img}}(f)\cos\left(2\pi ft\right)}{\pi f} df \\
&= \frac{\zeta}{2d_1^2 d_2^2} + \int_0^\infty \frac{f^4 + \varepsilon f^2 + \zeta}{\pi f \left(f^2 + d_1^2\right)\left(f^2 + d_2^2\right)} \sin\left(2\pi ft\right) df \\
&\quad - \int_0^\infty \frac{\nu f^2 + \tau}{\pi\left(f^2 + d_1^2\right)\left(f^2 + d_2^2\right)} \cos\left(2\pi ft\right) df .
\end{aligned} \tag{122}$$

This integration was performed numerically (in polar form, Eq. (63)), and gives the same result as the following calculation. By [Gradshteyn & Ryzhik, 1980, pp. 409, 3.728#4,#2], when f_1 and f_2 are unequal,

$$\begin{aligned}
&\int_0^\infty \frac{f^4 + \varepsilon f^2}{\pi f \left(f^2 + d_1^2\right)\left(f^2 + d_2^2\right)} \sin\left(2\pi ft\right) df \\
&\quad = \frac{d_1^2 \exp\left(-2\pi d_1 |t|\right) - d_2^2 \exp\left(-2\pi d_2 |t|\right)}{2\left(d_1^2 - d_2^2\right)} \operatorname{sgn}(t) \\
&\qquad + \varepsilon \frac{\exp\left(-2\pi d_1 |t|\right) - \exp\left(-2\pi d_2 |t|\right)}{2\left(d_2^2 - d_1^2\right)} \operatorname{sgn}(t).
\end{aligned} \tag{123}$$

Unfortunately the definite integral involving ζ is not listed in [Gradshteyn & Ryzhik, 1980], except when d_1 and d_2 are equal, in which case by [Gradshteyn & Ryzhik, 1980, pp. 412, 3.735#1]):

$$\begin{aligned}
&\int_0^\infty \frac{\zeta}{\pi f \left(f^2 + d_1^2\right)\left(f^2 + d_2^2\right)} \sin\left(2\pi ft\right) df \\
&\quad = \zeta \frac{1}{2d_1^2 d_2^2} \left[1 - \frac{1}{2} \exp\left(-2\pi d_1 |t|\right)\left(2 + 2\pi d_1 |t|\right)\right] \operatorname{sgn}(t).
\end{aligned} \tag{124}$$

Guessing the definite integral in the second part of the left hand side using the pattern of [Gradshteyn & Ryzhik, 1980, pp. 409, 3.728#1,#2,#3,#4] when d_1 and d_2 are unequal (we later used two other methods, numerical integration and estimation of the step response using the FFT, to verify the resulting step function, so we are confident that this guess is correct),

$$\int_0^\infty \frac{\zeta}{\pi f\left(f^2+d_1^2\right)\left(f^2+d_2^2\right)}\sin\left(2\pi ft\right)df$$
$$=\frac{\zeta}{2d_1^2d_2^2}\operatorname{sgn}(t)-\zeta\frac{f_2^{-2}\exp\left(-2\pi d_2|t|\right)-f_1^{-2}\exp\left(-2\pi d_1|t|\right)}{2\left(d_1^2-d_2^2\right)}\operatorname{sgn}(t). \tag{125}$$

By [Gradshteyn & Ryzhik, 1980, pp. 409, 3.728#3,#1], when d_1 and d_2 are unequal,

$$-\int_0^\infty \frac{\nu f^2+\tau}{\pi\left(f^2+d_1^2\right)\left(f^2+d_2^2\right)}\cos\left(2\pi ft\right)df$$
$$=-\nu\frac{d_1\exp\left(-2\pi d_1|t|\right)-d_2\exp\left(-2\pi d_2|t|\right)}{2\left(f_1^2-d_2^2\right)} \tag{126}$$
$$-\tau\frac{d_2^{-1}\exp\left(-2\pi d_2|t|\right)-d_1^{-1}\exp\left(-2\pi d_1|t|\right)}{2\left(f_1^2-d_2^2\right)}.$$

Thus

$$r_{\text{Notch},\mathbb{R}}(t)=\frac{\zeta}{d_1^2d_2^2}\operatorname{step}(t)+\frac{\exp\left(-2\pi d_1|t|\right)}{2\left(d_1^2-d_2^2\right)}\Lambda_1(t)+\frac{\exp\left(-2\pi d_2|t|\right)}{2\left(d_1^2-d_2^2\right)}\Lambda_2(t) \tag{127}$$

where, by Eq. (102),

$$\Lambda_1(t)=+d_1^2\operatorname{sgn}(t)-\varepsilon\operatorname{sgn}(t)+\zeta d_1^{-2}\operatorname{sgn}(t)-\nu d_1+\tau d_1^{-1}$$
$$\Lambda_2(t)=-d_2^2\operatorname{sgn}(t)+\varepsilon\operatorname{sgn}(t)-\zeta d_2^{-2}\operatorname{sgn}(t)+\nu d_2-\tau d_2^{-1}. \tag{128}$$

Let

$$\psi=2f_Z\cos\theta_Z. \tag{129}$$

Then by Eq. (121),

$$\Lambda_1(t)=\left[d_1^2-\varepsilon+\zeta d_1^{-2}\right]\operatorname{sgn}(t)-\nu d_1+\tau d_1^{-1}$$
$$=\left[d_1^2+f_Z^2+d_1d_2-(-1)^{k+l}\psi\left(d_1+d_2\right)+f_Z^2d_1d_1^{-1}\right]\operatorname{sgn}(t)$$
$$-(-1)^l\left(d_1+d_2\right)d_1+(-1)^k\psi d_1+(-1)^k\psi d_2-(-1)^l f_Z^2\left(d_1+d_2\right)d_1^{-1}$$
$$=\left[\left(d_1+d_2\right)d_1-(-1)^{k+l}\psi\left(d_1+d_2\right)+f_Z^2\left(d_1+d_2\right)d_1^{-1}\right]\operatorname{sgn}(t) \tag{130}$$
$$-(-1)^l\left(d_1+d_2\right)d_1+(-1)^k\psi\left(d_1+d_2\right)-(-1)^l f_Z^2\left(d_1+d_2\right)d_1^{-1}$$
$$=\left(d_1+d_2\right)\left[d_1-(-1)^{k+l}\psi+f_Z^2d_1^{-1}\right]\left[\operatorname{sgn}(t)-(-1)^l\right]$$

and

$$
\begin{aligned}
-\Lambda_2(t) &= \left[d_2^2 - \varepsilon + \zeta d_2^{-2} \right] \operatorname{sgn}(t) - \nu d_2 + \tau d_2^{-1} \\
&= \left[d_2^2 + f_Z^2 + d_1 d_2 - (-1)^{k+l} \psi \left(d_1 + d_2 \right) + f_Z^2 d_1 d_2^{-1} \right] \operatorname{sgn}(t) \\
&\quad - (-1)^l \left(d_1 + d_2 \right) d_2 + (-1)^k \psi d_2 + (-1)^k \psi d_1 - (-1)^l f_Z^2 \left(d_1 + d_2 \right) d_2^{-1} \\
&= \left[\left(d_1 + d_2 \right) d_2 - (-1)^{k+l} \psi \left(d_1 + d_2 \right) + f_Z^2 \left(d_1 + d_2 \right) d_2^{-1} \right] \operatorname{sgn}(t) \\
&\quad - (-1)^l \left(d_1 + d_2 \right) d_2 + (-1)^k \psi \left(d_1 + d_2 \right) - (-1)^l f_Z^2 \left(d_1 + d_2 \right) d_2^{-1} \\
&= \left(d_1 + d_2 \right) \left[d_2 - (-1)^{k+l} \psi + f_Z^2 d_2^{-1} \right] \left[\operatorname{sgn}(t) - (-1)^l \right].
\end{aligned}
\tag{131}
$$

Hence the step response of the 2nd order notch filter with real poles, whose transfer function is in Eq. (99), is

$$
r_{\mathrm{Notch},\mathbb{R}}(t) = \frac{f_Z^2}{d_1 d_2} \operatorname{step}(t) + \frac{1}{2} \left[\operatorname{sgn}(t) - (-1)^l \right] \left[A_{\mathbb{R}} \exp\left(-2\pi d_1 |t| \right) - B_{\mathbb{R}} \exp\left(-2\pi d_2 |t| \right) \right] \tag{132}
$$

where the constants (independent of t) are

$$
\begin{aligned}
A_{\mathbb{R}} &= \frac{d_1 - (-1)^{k+l}\, 2 f_Z \cos\theta_Z + f_Z^2 d_1^{-1}}{d_1 - d_2} \\
B_{\mathbb{R}} &= \frac{d_2 - (-1)^{k+l}\, 2 f_Z \cos\theta_Z + f_Z^2 d_2^{-1}}{d_1 - d_2}.
\end{aligned}
\tag{133}
$$

The character of the step response depends decisively on l. For real poles, if l is zero then the step response is

$$
r_{\mathrm{Notch},\mathbb{R},0}(t) = \begin{cases} -A_{\mathbb{R}} \exp\left(2\pi d_1 t \right) + B_{\mathbb{R}} \exp\left(2\pi d_2 t \right) & \text{if } t < 0 \\ \dfrac{1}{2} & \text{if } t = 0 \\ \dfrac{f_Z^2}{d_1 d_2} & \text{if } t > 0, \end{cases} \tag{134}
$$

which is non-causal. However if l is one then the step response is causal:

$$
r_{\mathrm{Notch},\mathbb{R},1}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{f_Z^2}{d_1 d_2} - \dfrac{1}{2} & \text{if } t = 0 \\ \dfrac{f_Z^2}{d_1 d_2} + A_{\mathbb{R}} \exp\left(-2\pi d_1 t \right) - B_{\mathbb{R}} \exp\left(-2\pi d_2 t \right) & \text{if } t > 0. \end{cases} \tag{135}
$$

The effect of k is confined to $A_{\mathbb{R}}$ and $B_{\mathbb{R}}$, which does not affect the character of the step response. l also affects $A_{\mathbb{R}}$ and $B_{\mathbb{R}}$.

This result has been checked against numerical integrations of Eq. (120) in Eq. (63). An example is shown in Fig. 14.

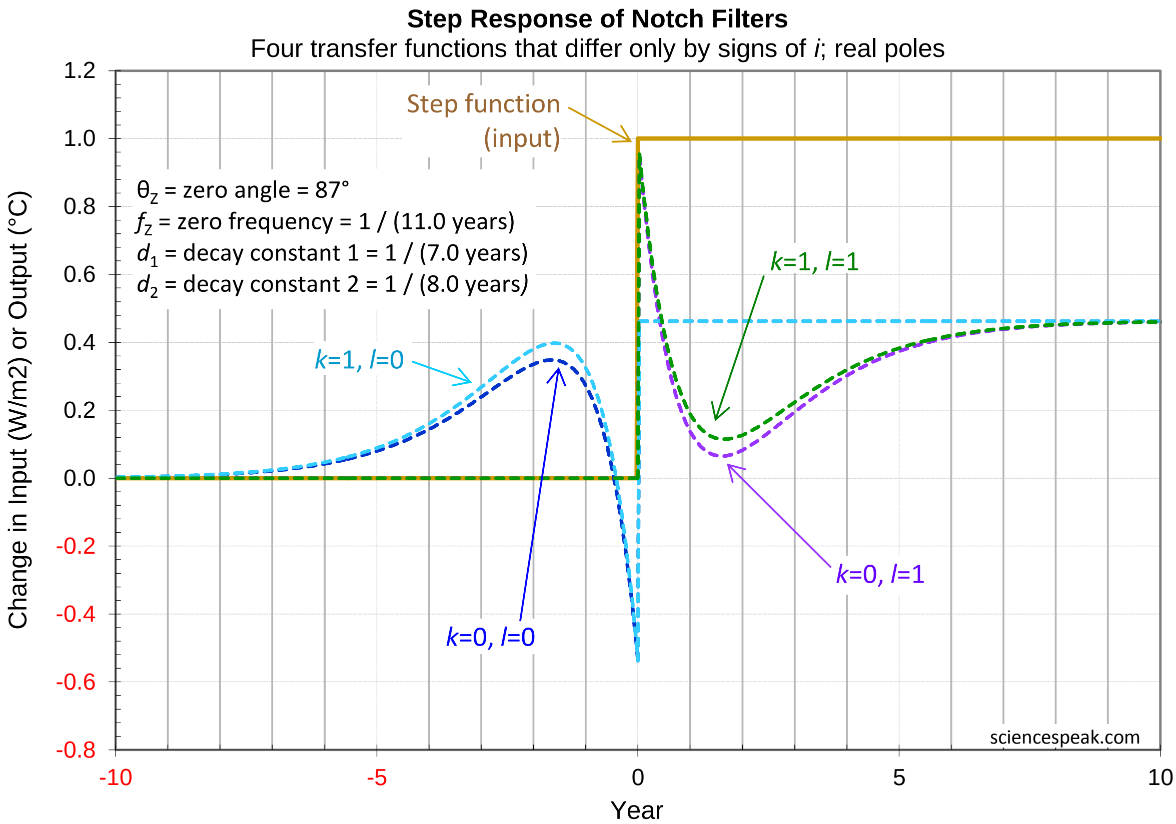


Figure 14: Step responses of the four notch filters in Fig. 12, with real poles. Similar to Fig. 13.

Fig.s 13 and 14 show that, at least for the parameter values depicted, the step response of a notch filter would be roughly causal (that is, "good enough") if it was simply delayed. At least to a good approximation, a delay of about 11 years would be sufficient.

10.4 Example 1: Series RLC Circuit

Here we examine a simple electronic circuit that implements a simple second-order notch filter. We find its transfer function from first principles, find its step response by applying the result based on Fourier-analysis above, then confirm from first principles that the step response so found is indeed its step response.

10.4.1 The System

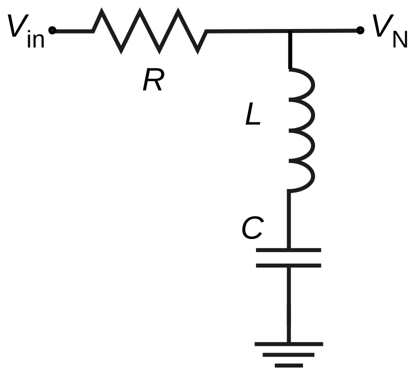


Figure 15: A series RLC circuit is a notch filter. It consists of a resistor *R*, inductor *L*, and capacitor *C* in series. The input voltage is V_{in}, and the output or notch voltage is V_N. The *LC* section is a resonant circuit, and near the resonant frequency its impedance is low. The notch voltage is just the voltage divider between the *R* and *LC* sections.

The input of the system is the applied voltage V_{in}, and the output is the notch voltage V_{N}. Let I denote the current in the circuit. Elementary circuit analysis yields two equations to describe the circuit:

$$L\frac{dI}{dt}+RI+\frac{1}{C}\int Idt=V_{\text{in}} \tag{136}$$

$$V_{\text{N}}=V_{\text{in}}-IR. \tag{137}$$

R, L, and C are positive and constant (independent of time), while V_{in}, I, and V_{N} are functions of time. All the components of the circuit are linear and invariant, so the system is a LIS.

For numerical expression in this example, let

$$R=3,\ L=1,\ C=1. \tag{138}$$

10.4.2 Differential Equation

The input voltage V_{in} is taken as given; we are trying to solve for the notch voltage V_{N}. Eliminate the intermediate variable I by substituting $R^{-1}\left(V_{\text{in}}-V_{\text{N}}\right)$ into Eq. (136), giving the single circuit equation

$$\frac{L}{R}\frac{dV_{\text{N}}}{dt}+V_{\text{N}}+\frac{1}{RC}\int V_{\text{N}}dt=\frac{L}{R}\frac{dV_{\text{in}}}{dt}+\frac{1}{RC}\int V_{\text{in}}dt. \tag{139}$$

Differentiating with respect to time and multiplying by R/L,

$$\left[D^2+\frac{R}{L}D+\frac{1}{LC}\right]V_{\text{N}}=\left[D^2+\frac{1}{LC}\right]V_{\text{in}} \tag{140}$$

where D is the differentiation operator d/dt. This last circuit equation is the linear differential equation that fully describes the system. It is linear and invariant, so it describes a LIS. It is a second-order linear non-homogeneous ordinary differential equation, whose methods for solution are well known, e.g. [Tseng, 2008]. The corresponding homogeneous equation is

$$\left[D^2+\frac{R}{L}D+\frac{1}{LC}\right]V_{\text{N}}=0, \tag{141}$$

whose characteristic equation is

$$D^2+\frac{R}{L}D+\frac{1}{LC}=0. \tag{142}$$

The difference between any two solutions to Eq. (140) is a solution to Eq. (141); therefore every solution to Eq. (140) is any “particular solution” to Eq. (140) plus one of the solutions to Eq. (141) (which are collectively called the “complementary solution”). In an electrical circuit, the complementary solution describes the transients (which hopefully die away to zero before long), and the particular solution is the steady-state solution [Edminister, 1965, p. 242].

The solution to the characteristic equation is

$$D = \alpha \pm \beta \tag{143}$$

where

$$\alpha = -\frac{R}{2L}, \quad \beta = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = \sqrt{\alpha^2 - \frac{1}{LC}}\,. \tag{144}$$

The solution has three cases, depending on whether β is the square root of a positive, zero, or negative real number—the overdamped, critically damped, and underdamped cases respectively. In what follows we need only consider one of the main cases, so we are only going to consider the overdamped case; thus β is a real number and

$$\alpha < 0, \quad \beta > 0, \quad |\alpha| > \beta, \quad \alpha \pm \beta < 0\,. \tag{145}$$

Continuing our numerical expression,

$$\alpha = -\frac{3}{2} = -1.5, \quad \beta = \frac{\sqrt{5}}{2} \simeq 1.12, \quad \alpha \pm \beta = \frac{-3 \pm \sqrt{5}}{2} \simeq -0.38, -2.62\,. \tag{146}$$

The differential equation is expressed in terms of α and β as

$$\left[D - (\alpha + \beta)\right]\left[D - (\alpha - \beta)\right] V_{\mathrm{N}} = \left[D^2 - 2\alpha D + \left(\alpha^2 - \beta^2\right)\right] V_{\mathrm{N}} = \left[D^2 + \frac{1}{LC}\right] V_{\mathrm{in}}\,, \tag{147}$$

while the homogeneous equation is

$$\left[D - (\alpha + \beta)\right]\left[D - (\alpha - \beta)\right] V_{\mathrm{N}} = \left[D^2 - 2\alpha D + \left(\alpha^2 - \beta^2\right)\right] V_{\mathrm{N}} = 0\,. \tag{148}$$

Because the roots of the characteristic equation (Eq. (142)) are real and are $\alpha \pm \beta$, the complementary solution (the solution to the homogeneous equation) is

$$V_{\mathrm{N,C}}(t) = c_1 e^{(\alpha+\beta)t} + c_2 e^{(\alpha-\beta)t} \tag{149}$$

for some constants $c_1, c_2 \in \mathbb{R}$ that are determined by the boundary conditions after the general solution (the sum of a particular solution and the complementary solution) is found. Note that both terms in the complementary solution each satisfy the homogeneous equation:

$$\left[D - (\alpha \pm \beta)\right] c_j e^{(\alpha \pm \beta)t} = 0\,. \tag{150}$$

Because $\alpha \pm \beta$ are negative, the complementary solution consists of two transients that die away with time, but which are infinite for infinitely negative times.

10.4.3 Transfer function

When we excite the system with an input voltage V_{in} that is a sinusoid at frequency f_{e}, the output V_{N} will also be a sinusoid at f_{e} because the system is a LIS. The value of the system's transfer function at f_{e} is the output sinusoid represented as a complex number (as per Eq. (18)) divided by the input sinusoid represented as a complex number.

Let the input voltage V_{in} be $\cos(\omega_{\text{e}}t)$, where $\omega_{\text{e}} = 2\pi f_{\text{e}}$ and $f_{\text{e}} > 0$. It is represented by the complex number unity, so the transfer function is just the complex number that represents the output sinusoid. The differential equation describing the system (Eq. (147)) becomes

$$\left[D^2 - 2\alpha D + \left(\alpha^2 - \beta^2\right)\right]V_{\text{N}} = -\omega_{\text{e}}^2 \cos(\omega_{\text{e}}t) + \frac{1}{LC}\cos(\omega_{\text{e}}t)\,. \tag{151}$$

V_{N} is a sinusoid at frequency f_{e}, so let us try a particular solution of the form

$$V_{\text{N}}(t) = A\cos(\omega_{\text{e}}t) + B\sin(\omega_{\text{e}}t) \tag{152}$$

for some real numbers *A* and *B*. Then the LHS of Eq. (151) becomes

$$\begin{aligned}
&\left[D^2 - 2\alpha D + \left(\alpha^2 - \beta^2\right)\right]\left[A\cos(\omega_{\text{e}}t) + B\sin(\omega_{\text{e}}t)\right] \\
&\qquad = -\omega_{\text{e}}^2\left[A\cos(\omega_{\text{e}}t) + B\sin(\omega_{\text{e}}t)\right] \\
&\qquad\quad - 2\alpha\omega_{\text{e}}\left[-A\sin(\omega_{\text{e}}t) + B\cos(\omega_{\text{e}}t)\right] \\
&\qquad\quad + \left(\alpha^2 - \beta^2\right)\left[A\cos(\omega_{\text{e}}t) + B\sin(\omega_{\text{e}}t)\right] \\
&\qquad = \left[-\omega_{\text{e}}^2 A - 2\alpha\omega_{\text{e}}B + \left(\alpha^2 - \beta^2\right)A\right]\cos(\omega_{\text{e}}t) \\
&\qquad\quad + \left[-\omega_{\text{e}}^2 B + 2\alpha\omega_{\text{e}}A + \left(\alpha^2 - \beta^2\right)B\right]\sin(\omega_{\text{e}}t)
\end{aligned} \tag{153}$$

so

$$\begin{aligned}
&-\omega_{\text{e}}^2 A - 2\alpha\omega_{\text{e}}B + \left(\alpha^2 - \beta^2\right)A = -\omega_{\text{e}}^2 + \frac{1}{LC} \\
&-\omega_{\text{e}}^2 B + 2\alpha\omega_{\text{e}}A + \left(\alpha^2 - \beta^2\right)B = 0.
\end{aligned} \tag{154}$$

Hence

$$B = \frac{-2\alpha\omega_{\text{e}}}{\alpha^2 - \beta^2 - \omega_{\text{e}}^2}A \tag{155}$$

and

$$\begin{aligned}
A &= \frac{2\alpha\omega_{\text{e}}B + L^{-1}C^{-1} - \omega_{\text{e}}^2}{\alpha^2 - \beta^2 - \omega_{\text{e}}^2} \\
&= \frac{2\alpha\omega_{\text{e}}\dfrac{-2\alpha\omega_{\text{e}}}{\alpha^2 - \beta^2 - \omega_{\text{e}}^2}A + L^{-1}C^{-1} - \omega_{\text{e}}^2}{\alpha^2 - \beta^2 - \omega_{\text{e}}^2} \\
&= \frac{-4\alpha^2\omega_{\text{e}}^2 A + \left(L^{-1}C^{-1} - \omega_{\text{e}}^2\right)\left(\alpha^2 - \beta^2 - \omega_{\text{e}}^2\right)}{\left(\alpha^2 - \beta^2 - \omega_{\text{e}}^2\right)^2}
\end{aligned} \tag{156}$$

so

$$A = \lambda\left(\alpha^2 - \beta^2 - \omega_{\text{e}}^2\right) \tag{157}$$

and

$$B = -2\alpha\omega_{\mathrm{e}}\lambda \tag{158}$$

where

$$\lambda = \frac{L^{-1}C^{-1} - \omega_{\mathrm{e}}^2}{4\alpha^2\omega_{\mathrm{e}}^2 + \left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)^2}\,. \tag{159}$$

Therefore a particular solution is

$$V_{\mathrm{N}}(t) = \lambda\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t) - 2\alpha\omega_{\mathrm{e}}\lambda\sin(\omega_{\mathrm{e}}t)\,. \tag{160}$$

[Check by substituting it into Eq. (151):

$$\begin{aligned}
LHS &= \left[D^2 - 2\alpha D + \left(\alpha^2 - \beta^2\right)\right]V_{\mathrm{N}} \\
&= \lambda\left\{\begin{aligned}
&-\omega_{\mathrm{e}}^2\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t) + 2\alpha\omega_{\mathrm{e}}^3\sin(\omega_{\mathrm{e}}t) \\
&+2\alpha\omega_{\mathrm{e}}\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\sin(\omega_{\mathrm{e}}t) + 4\alpha^2\omega_{\mathrm{e}}^2\cos(\omega_{\mathrm{e}}t) \\
&+\left(\alpha^2 - \beta^2\right)\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t) - 2\alpha\omega_{\mathrm{e}}\left(\alpha^2 - \beta^2\right)\sin(\omega_{\mathrm{e}}t)
\end{aligned}\right\} \\
&= \lambda\left\{-\omega_{\mathrm{e}}^2\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right) + 4\alpha^2\omega_{\mathrm{e}}^2 + \left(\alpha^2 - \beta^2\right)\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\right\}\cos(\omega_{\mathrm{e}}t) \\
&\quad +\lambda\left\{+2\alpha\omega_{\mathrm{e}}^3 + 2\alpha\omega_{\mathrm{e}}\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right) - 2\alpha\omega_{\mathrm{e}}\left(\alpha^2 - \beta^2\right)\right\}\sin(\omega_{\mathrm{e}}t) \\
&= \lambda\left\{4\alpha^2\omega_{\mathrm{e}}^2 + \left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\right\}\cos(\omega_{\mathrm{e}}t) \\
&\quad +\lambda\left\{0\right\}\sin(\omega_{\mathrm{e}}t) \\
&= \left(L^{-1}C^{-1} - \omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t) \\
&= RHS\,.\ ]
\end{aligned}$$

Adding the complementary solution in Eq. (149), the general solution is

$$V_{\mathrm{N}}(t) = \lambda\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t) - 2\alpha\omega_{\mathrm{e}}\lambda\sin(\omega_{\mathrm{e}}t) + c_1 e^{(\alpha+\beta)t} + c_2 e^{(\alpha-\beta)t} \tag{161}$$

for some $c_1, c_2 \in \mathbb{R}$. The boundary condition that V_{N} remains finite as time goes to positive or negative infinity implies that c_1 and c_2 are zero—which is to be expected because these represent transients but a true transfer function is the response to an input sinusoid that exists for all time and thus involves no transients. The output is therefore

$$V_{\mathrm{N}}(t) = \lambda\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t) - 2\alpha\omega_{\mathrm{e}}\lambda\sin(\omega_{\mathrm{e}}t)\,, \tag{162}$$

the sinusoid at frequency f_{e} represented by the complex number $\lambda\left(\alpha^2 - \beta^2 - \omega_{\mathrm{e}}^2\right) - i\,2\alpha\omega_{\mathrm{e}}\lambda$. By Eq. (144), $\alpha^2 - \beta^2$ is equal to $L^{-1}C^{-1}$. Hence the transfer function of the system is

$$H_{\mathrm{N}}(f) = \frac{\left(\alpha^2 - \beta^2 - 4\pi^2 f^2\right) - i\,4\pi\alpha f}{16\pi^2\alpha^2 f^2 + \left(\alpha^2 - \beta^2 - 4\pi^2 f^2\right)^2}\left(\alpha^2 - \beta^2 - 4\pi^2 f^2\right), \quad f \geq 0\,. \tag{163}$$

Observe that this is consistent with a notch filter:

- As $f \to 0$, $\left|H_{\mathrm{N}}(f)\right| \to 1$.

- As $f \to \infty$, $|H_{\mathrm{N}}(f)| \to 1$, the same non-zero value as when $f \to 0$.
- $|H_{\mathrm{N}}(f)|$ is positive for all *f*, except that it is zero when *f* is $1/2\pi\sqrt{LC}$. Thus it is a minimum at $1/2\pi\sqrt{LC}$, which is the notch frequency.

We can establish if this is a notch filter by seeing if it is of the form of the transfer function of a second-order notch filter, above. The solutions of the characteristic equation (Eq. (142)) are real, so if it was a notch filter it would have real poles rather than complex poles and its transfer function would be like $H_{\mathrm{Notch},\mathbb{R}}$. We need to compare $H_{\mathrm{Notch},\mathbb{R}}$ in Eq. (120) with H_{N} in Eq. (163) in order to find ε, ζ, ν, and τ, and then from those find the f_{Z}, θ_{Z}, d_1, and d_2 parameters used in Eq. (99). Let's start by matching the denominators. The denominator of H_{N} in Eq. (163) is

$$\begin{aligned}
&16\pi^2\alpha^2 f^2 + \left(\alpha^2 - \beta^2 - 4\pi^2 f^2\right)^2 \\
&\qquad = 16\pi^2\alpha^2 f^2 + \left(\alpha^2 - \beta^2\right)^2 + 16\pi^4 f^4 - 8\pi^2 f^2\left(\alpha^2 - \beta^2\right) \\
&\qquad = 16\pi^4 f^4 + 8\pi^2 f^2\left(\alpha^2 + \beta^2\right) + \left(\alpha^2 - \beta^2\right)^2 \\
&\qquad = 16\pi^4\left\{ f^4 + \frac{f^2\left(\alpha^2+\beta^2\right)}{2\pi^2} + \frac{\left(\alpha^2-\beta^2\right)^2}{16\pi^4} \right\},
\end{aligned}$$

and setting this to the denominator of $H_{\mathrm{Notch},\mathbb{R}}$ in Eq. (120) gives

$$f^4 + \frac{f^2\left(\alpha^2+\beta^2\right)}{2\pi^2} + \frac{\left(\alpha^2-\beta^2\right)^2}{16\pi^4} = \left(f^2 + d_1^2\right)\left(f^2 + d_2^2\right). \tag{164}$$

Therefore

$$\begin{aligned}
d_1^2 + d_2^2 &= \frac{\alpha^2+\beta^2}{2\pi^2} \\
d_1^2 d_2^2 &= \frac{\left(\alpha^2-\beta^2\right)^2}{16\pi^4},
\end{aligned} \tag{165}$$

so

$$d_1^2 = \left(\frac{\alpha+\beta}{2\pi}\right)^2,\ d_2^2 = \left(\frac{\alpha-\beta}{2\pi}\right)^2. \tag{166}$$

Because d_1 and d_2 are positive (Eq. (100)) while $\alpha \pm \beta$ are negative (Eq. (145)), only the negative square roots are valid:

$$d_1 = -\frac{\alpha+\beta}{2\pi},\ d_2 = -\frac{\alpha-\beta}{2\pi}. \tag{167}$$

Thus

$$d_1 + d_2 = -\frac{\alpha}{\pi},\ d_1 - d_2 = -\frac{\beta}{\pi},\ d_1 d_2 = \frac{\alpha^2-\beta^2}{4\pi^2}. \tag{168}$$

Now match the numerators. With the denominators as in Eq. (164), the numerator of H_{N} is

$$\frac{\left(\alpha^2-\beta^2-4\pi^2 f^2\right)-i\,4\pi\alpha f}{16\pi^4}\left(\alpha^2-\beta^2-4\pi^2 f^2\right) = \frac{\left(\alpha^2-\beta^2-4\pi^2 f^2\right)^2}{16\pi^4}-i\frac{\alpha f\left(\alpha^2-\beta^2-4\pi^2 f^2\right)}{4\pi^3}, \tag{169}$$

while the numerator of $H_{\mathrm{Notch},\mathbb{R}}$ is $\left(f^4+\varepsilon f^2+\zeta\right)+i\left(\nu f^3+\tau f\right)$. Thus, substituting for α and β using Eq. (168),

$$\begin{aligned}
\varepsilon &= -\frac{\alpha^2-\beta^2}{2\pi^2} = -2d_1 d_2 \\
\zeta &= \frac{\left(\alpha^2-\beta^2\right)^2}{16\pi^4} = \left(d_1 d_2\right)^2 \\
\nu &= \frac{\alpha}{\pi} = -d_1-d_2 \\
\tau &= -\frac{\alpha\left(\alpha^2-\beta^2\right)}{4\pi^3} = -\frac{\alpha d_1 d_2}{\pi} = \left(d_1+d_2\right)d_1 d_2.
\end{aligned} \tag{170}$$

We now compare these to Eq. (121). From the equation for ζ, f_{Z}^2 is equal to $d_1 d_2$. Substituting for f_{Z}^2 in the equation for ε, $\cos\theta_{\mathrm{Z}}$ is zero so θ_{Z} is 90°. Substituting for $\cos\theta_{\mathrm{Z}}$ in the equation for ν, l is one. Substituting for f_{Z}^2, $\cos\theta_{\mathrm{Z}}$, and l satisfies the equation for τ. Thus the mapping between H_{N} and $H_{\mathrm{Notch},\mathbb{R}}$ is:

$$d_1 = -\frac{\alpha+\beta}{2\pi},\;\; d_2 = -\frac{\alpha-\beta}{2\pi},\;\; f_{\mathrm{Z}}^2 = d_1 d_2 = \frac{\alpha^2-\beta^2}{4\pi^2},\;\; \theta_{\mathrm{Z}} = 90°,\;\; l = 1. \tag{171}$$

The zeroes and poles are (see Eq.s (92), (171), (144), (97))

$$\begin{aligned}
z_1, z_2 &= 2\pi f_{\mathrm{Z}}\left(-\cos\theta_{\mathrm{Z}} \pm i\sin\theta_{\mathrm{Z}}\right) = \pm i\,2\pi f_{\mathrm{Z}} = \pm i\,2\pi\sqrt{d_1 d_2} = \pm i\sqrt{\alpha^2-\beta^2} = \pm i\frac{1}{\sqrt{LC}} \\
p_1, p_2 &= -2\pi d_1, -2\pi d_2 = \alpha \pm \beta = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}.
\end{aligned} \tag{172}$$

Continuing our numerical expression,

$$d_1, d_2 = -\frac{\alpha\pm\beta}{2\pi} = -\frac{-3\pm\sqrt{5}}{4\pi} = 0.061, 0.417 \text{ Hz},\quad f_{\mathrm{Z}} = \frac{1}{2\pi} \simeq 0.159 \text{ Hz},\quad \theta_{\mathrm{Z}} = 90° \tag{173}$$

and

$$\begin{aligned}
z_1, z_2 &= \pm i \text{ rad/s} = \pm\frac{i}{2\pi}\text{ Hz} \simeq \pm i\,0.159 \text{ Hz} \\
p_1, p_2 &= -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 1} \simeq -0.38, -2.62 \text{ rad/s} \simeq -0.061, -0.417 \text{ Hz},
\end{aligned} \tag{174}$$

which agree with industry website [Okawa Electric Design, 2015].

That the transfer function of the system in Eq. (163) can be mapped onto the transfer function of a notch filter in Eq. (120) confirms that the system is indeed a notch filter. We can use that mapping to give the transfer function in the form of Eq. (99):

$$\begin{aligned} H_{\mathrm{N}}(f) &= \frac{f_{\mathrm{Z}}^2 - f^2}{d_1 d_2 - f^2 - i f\left(d_1 + d_2\right)} \\ &= \frac{\alpha^2 - \beta^2 - 4\pi^2 f^2}{\alpha^2 - \beta^2 - 4\pi^2 f^2 + i 4\pi\alpha f} \\ &= \frac{1 - 4\pi^2 LCf^2}{1 - 4\pi^2 LCf^2 - i2\pi RCf}. \end{aligned} \tag{175}$$

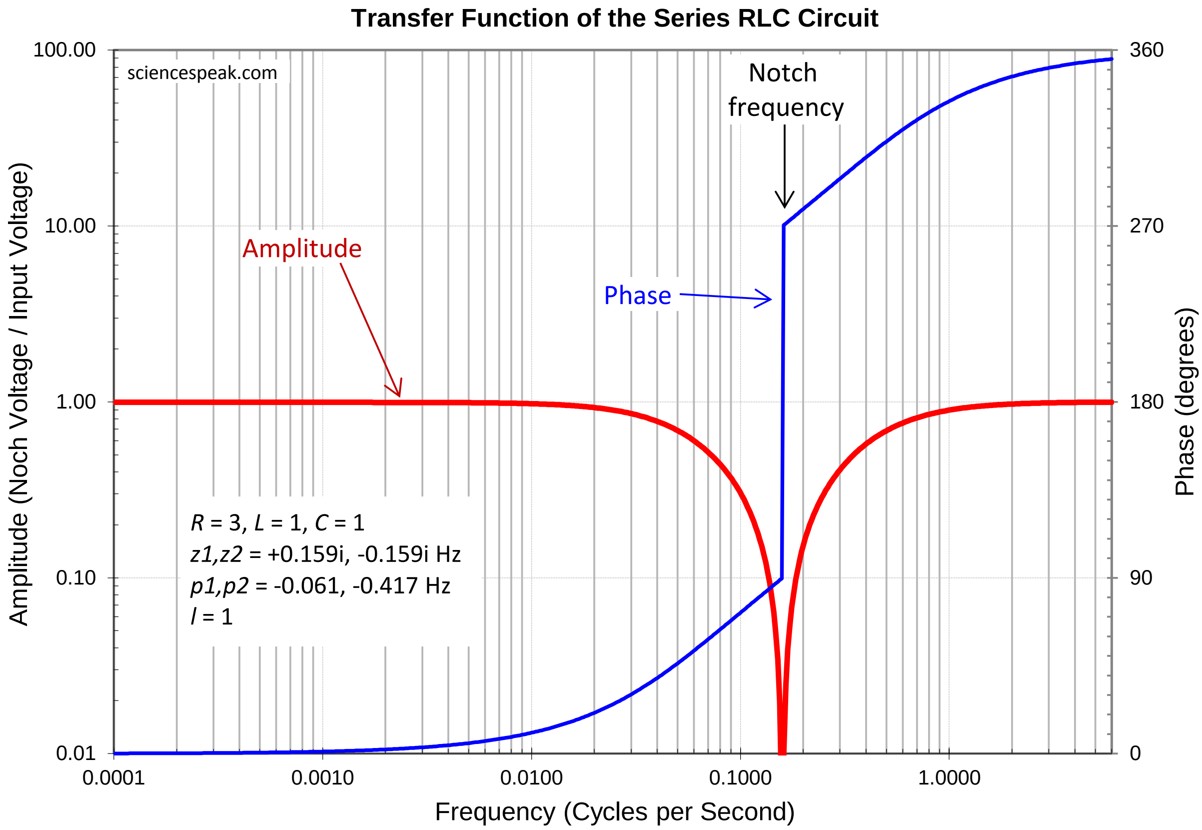


Figure 16: Transfer function of the series RLC circuit. Note the notch in the amplitude.

10.4.4 Step Response from the Transfer Function

Now that we have the transfer function of the system in the form used to compute the step response by Fourier analysis above, we can apply that result. Substituting the mapping of Eq. (171) into the step response in Eq.s (132) and (133), the system's step response is

$$r(t) = \mathrm{step}(t) + \frac{\alpha}{\beta}\mathrm{step}(t)\left[e^{(\alpha+\beta)t} - e^{(\alpha-\beta)t}\right], \tag{176}$$

which is causal, with the transients in the second term only existing when t is positive and dying away as t increases. (k was not determined above, but it does not affect the step response because $\cos\theta_{\mathrm{Z}}$ is zero. Thus the Fourier analysis gave only this one step response.)

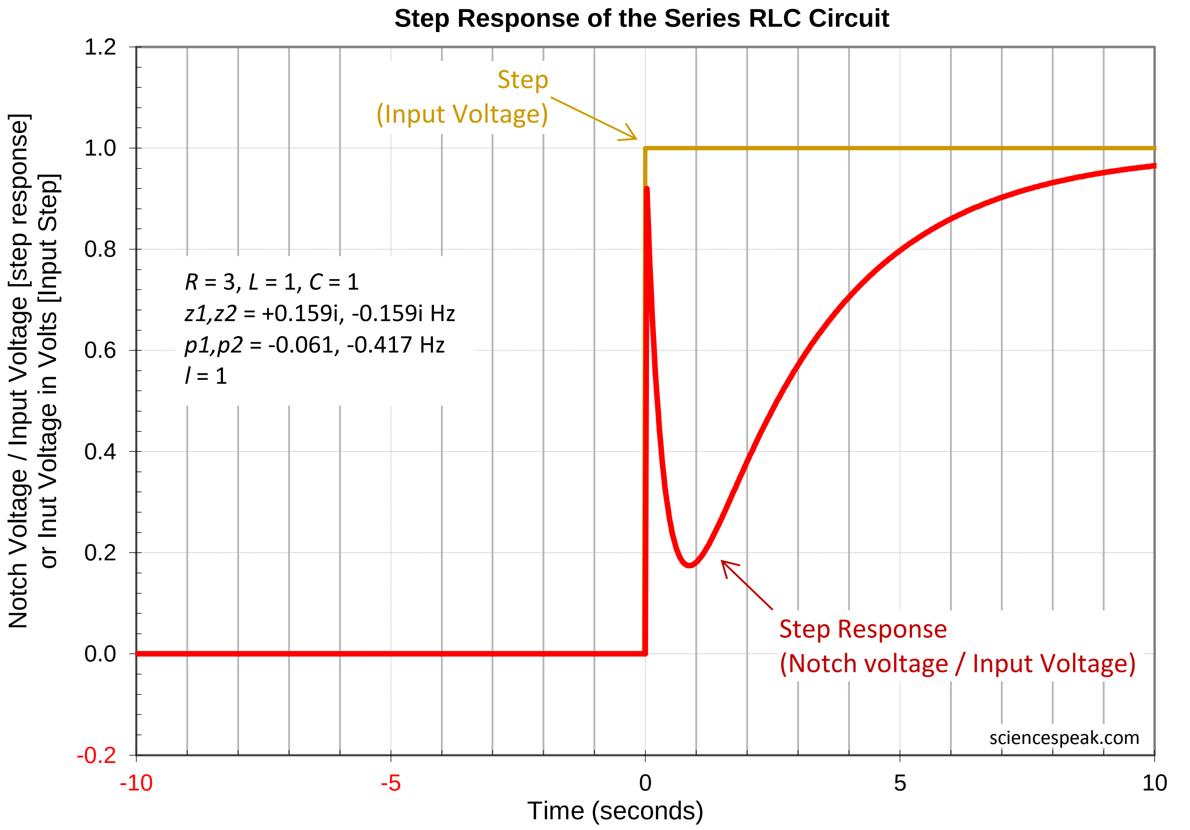


Figure 17: Step response of the series RLC circuit. It is causal. The response is the same as the step input, except for some transients that soon die out.

10.4.5 System Stability

The transfer function in Eq. (175) can be converted to the Laplace-transform transfer function by substituting s for $-i2\pi f$:

$$H_{\mathrm{N,s}}(s) = \frac{1+s^2LC}{1+s^2LC+sRC} = \frac{s^2+\dfrac{1}{LC}}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}}. \tag{177}$$

The complex frequency s is $\sigma + i2\pi f$, where σ is the rate constant in an exponential factor e^{st} that multiplies the sinusoids in a Fourier analysis as performed by a Laplace transform. In (regular) Fourier analysis as above, σ is zero. Here s is $-i2\pi f$ rather than $+i2\pi f$ because l is one—see Eq. (83). Electrical engineers will recognize $H_{\mathrm{N,s}}$ as the expression for $V_{\mathrm{N}}/V_{\mathrm{in}}$ in the Laplace-transformed version of Fig. 15. The poles of $H_{\mathrm{N,s}}$ are the solutions of the characteristic equation, namely $\alpha \pm \beta$, which are both negative and thus in the left half of the s-plane—so the system is stable. Alternatively note that the transients of the step response in Eq. (176) get smaller as time progresses—so the system is stable.

10.4.6 Step Response by Solving the Differential Equation

Here we confirm from first principles that the function r in Eq. (176) is the step response R of the system, by testing r in the differential equation for the system. The step response R is the output voltage V_{N} when the input voltage V_{in} is the step function, so it must satisfy Eq. (147):

$$\left[D^2-2\alpha D+\left(\alpha^2-\beta^2\right)\right]R=\left[D^2+L^{-1}C^{-1}\right]\text{step}. \tag{178}$$

The step function has derivatives that involve delta functions; let us denote them by

$$\begin{aligned} D\,\text{step}&=\text{step}' \\ D^2\,\text{step}&=\text{step}''. \end{aligned} \tag{179}$$

We now substitute r in Eq. (176) into Eq. (178). First, note that

$$\left[D^2-2\alpha D+\left(\alpha^2-\beta^2\right)\right]\text{step}(t)=\text{step}''(t)-2\alpha\,\text{step}'(t)+\left(\alpha^2-\beta^2\right)\text{step}(t) \tag{180}$$

and, because step' and step'' are zero except when their argument is zero,

$$\begin{aligned} &\left[D^2-2\alpha D+\left(\alpha^2-\beta^2\right)\right]\text{step}(t)e^{(\alpha\pm\beta)t} \\ &\quad=\left[\text{step}''(t)e^{(\alpha\pm\beta)t}+\left(\alpha\pm\beta\right)\text{step}'(t)e^{(\alpha\pm\beta)t}+\text{step}(t)\left(\alpha\pm\beta\right)^2e^{(\alpha\pm\beta)t}\right] \\ &\quad\quad-2\alpha\left[\text{step}'(t)e^{(\alpha\pm\beta)t}+\text{step}(t)\left(\alpha\pm\beta\right)e^{(\alpha\pm\beta)t}\right] \\ &\quad\quad+\left(\alpha^2-\beta^2\right)\text{step}(t)e^{(\alpha\pm\beta)t} \\ &\quad=e^{(\alpha+\beta)t}\begin{Bmatrix}\left[+\left(\alpha\pm\beta\right)^2-2\alpha\left(\alpha\pm\beta\right)+\left(\alpha^2-\beta^2\right)\right]\text{step}(t) \\ +\left[\left(\alpha\pm\beta\right)-2\alpha\right]\text{step}'(t)+\text{step}''(t)\end{Bmatrix} \\ &\quad=e^{(\alpha+\beta)t}\begin{Bmatrix}\left[\alpha^2\pm2\alpha\beta+\beta^2-2\alpha^2\mp2\alpha\beta+\alpha^2-\beta^2\right]\text{step}(t) \\ +\left[\beta-\alpha\right]\text{step}'(t)+\text{step}''(t)\end{Bmatrix} \\ &\quad=\left(-\alpha\pm\beta\right)\text{step}'(t)+\text{step}''(t). \end{aligned} \tag{181}$$

Then, with the aid of Eq. (144),

$$\begin{aligned} &\left[D^2-2\alpha D+\left(\alpha^2-\beta^2\right)\right]r(t) \\ &\quad=\left[D^2-2\alpha D+\left(\alpha^2-\beta^2\right)\right]\left\{\text{step}(t)+\frac{\alpha}{\beta}\text{step}(t)\left[e^{(\alpha+\beta)t}-e^{(\alpha-\beta)t}\right]\right\} \\ &\quad=\text{step}''(t)-2\alpha\,\text{step}'(t)+\left(\alpha^2-\beta^2\right)\text{step}(t) \\ &\quad\quad+\frac{\alpha}{\beta}\left[\left(-\alpha+\beta\right)\text{step}'(t)+\text{step}''(t)-\left(-\alpha-\beta\right)\text{step}'(t)-\text{step}''(t)\right] \\ &\quad=\text{step}''(t)+\left(\alpha^2-\beta^2\right)\text{step}(t) \\ &\quad=\left[D^2+L^{-1}C^{-1}\right]\text{step}(t) \end{aligned}$$

so the function r in Eq. (176) is indeed the step response of the system.

10.5 Example 2: Reversed Series RLC Circuit

Consider an imaginary system that is the same as the notch filter circuit in Section 10.4, except that time runs backwards! We do this merely to generate a system that is the non-causal counterpart of the previous example, not because time really runs backwards or we have any interest in knowing what might happen if could be said to run backwards.

10.5.1 The System

The circuit equation is as per Eq. (139) except that dt becomes $-dt$, so the equation describing this system is

$$-\frac{L}{R}\frac{dV}{dt}-\frac{1}{RC}\int V dt=-\frac{L}{R}\frac{dV_{\mathrm{N}}}{dt}+V_{\mathrm{N}}-\frac{1}{RC}\int V_{\mathrm{N}} dt\,. \tag{182}$$

10.5.2 Differential Equation

Differentiating with respect to time and multiplying by $-R/L$,

$$\left[D^2-\frac{R}{L}D+\frac{1}{LC}\right]V_{\mathrm{N}}=\left[D^2+\frac{1}{LC}\right]V_{\mathrm{in}}\,. \tag{183}$$

This circuit equation is the linear differential equation that fully describes the system, differing from Eq. (140) only in the sign of the lone D term. Its homogeneous equation is

$$\left[D^2-\frac{R}{L}D+\frac{1}{LC}\right]V_{\mathrm{N}}=0\,, \tag{184}$$

whose solution is

$$D=-\alpha\pm\beta \tag{185}$$

where α and β are as above in Eq.s (144) – (146). The differential equation expressed in terms of α and β is thus

$$\big[D-(-\alpha+\beta)\big]\big[D-(-\alpha-\beta)\big]V_{\mathrm{N}}=\big[D^2+2\alpha D+(\alpha^2-\beta^2)\big]V_{\mathrm{N}}=\left[D^2+\frac{1}{LC}\right]V_{\mathrm{in}}\,, \tag{186}$$

again differing from Eq. (147) only in the sign of the D term.

10.5.3 Transfer function

To find the system's transfer function we excite it with input $\cos(\omega_{\mathrm{e}}t)$ as above, whereupon the output V_{N} must satisfy

$$\big[D^2+2\alpha D+(\alpha^2-\beta^2)\big]V_{\mathrm{N}}=-\omega_{\mathrm{e}}^2\cos(\omega_{\mathrm{e}}t)+\frac{1}{LC}\cos(\omega_{\mathrm{e}}t)\,. \tag{187}$$

Proceeding as in section 10.4.3, the solution to this is

$$V_{\mathrm{N}}(t)=\lambda\left(\alpha^2-\beta^2-\omega_{\mathrm{e}}^2\right)\cos(\omega_{\mathrm{e}}t)+2\alpha\omega_{\mathrm{e}}\lambda\sin(\omega_{\mathrm{e}}t)\,, \tag{188}$$

which is the same as Eq. (162) except that the sign of the sine is flipped. Hence the transfer function of the system is

$$H_{\mathrm{N}*}(f)=\frac{\left(\alpha^2-\beta^2-4\pi^2f^2\right)+i\,4\pi\alpha f}{16\pi^2\alpha^2f^2+\left(\alpha^2-\beta^2-4\pi^2f^2\right)^2}\left(\alpha^2-\beta^2-4\pi^2f^2\right),\quad f\geq 0\,, \tag{189}$$

which is identical to H_{N} in Eq. (163) except that the sign of *i* is flipped. To map this to $H_{\mathrm{Notch},\mathbb{R}}$ in Eq. (120), we proceed as before up to Eq. (169), which is the same except the sign of *i* is flipped. Then, flipping the sign of ν and τ, Eq. (170) becomes

$$\begin{aligned}
\varepsilon &= -\frac{\alpha^2-\beta^2}{2\pi^2} = -2d_1d_2 \\
\zeta &= \frac{\left(\alpha^2-\beta^2\right)^2}{16\pi^4} = \left(d_1d_2\right)^2 \\
\nu &= -\frac{\alpha}{\pi} = d_1+d_2 \\
\tau &= \frac{\alpha\left(\alpha^2-\beta^2\right)}{4\pi^3} = \frac{\alpha d_1d_2}{\pi} = -\left(d_1+d_2\right)d_1d_2.
\end{aligned} \tag{190}$$

Now when we compare these to Eq. (121), the equation for ζ implies f_{Z}^2 is equal to d_1d_2 and the equation for ε implies θ_{Z} is 90°, as above. Substituting for $\cos\theta_{\mathrm{Z}}$ in the equation for ν, *l* is now zero instead. Substituting for f_{Z}^2, $\cos\theta_{\mathrm{Z}}$, and *l* satisfies the equation for τ. Thus the mapping between $H_{\mathrm{N*}}$ and $H_{\mathrm{Notch},\mathbb{R}}$ is:

$$d_1 = -\frac{\alpha+\beta}{2\pi},\ \ d_2 = -\frac{\alpha-\beta}{2\pi},\ \ f_{\mathrm{Z}}^2 = d_1d_2 = \frac{\alpha^2-\beta^2}{4\pi^2},\ \ \theta_{\mathrm{Z}} = 90°,\ \ l = 0, \tag{191}$$

which is as per Eq. (171) but *l* is zero instead of one. The zeroes and poles we are using to characterize this transfer function are as in the previous example, Eq.s (172) – (174). However, by this mapping, the system's transfer function in the form of Eq. (99) is identical to H_{N} in Eq. (175) except that the sign of *i* is flipped:

$$\begin{aligned}
H_{\mathrm{N*}}(f) &= \frac{f_{\mathrm{Z}}^2 - f^2}{d_1d_2 - f^2 + i f\left(d_1+d_2\right)} \\
&= \frac{\alpha^2-\beta^2-4\pi^2 f^2}{\alpha^2-\beta^2-4\pi^2 f^2 - i4\pi\alpha f} \\
&= \frac{1-4\pi^2 LCf^2}{1-4\pi^2 LCf^2 + i2\pi RCf}.
\end{aligned} \tag{192}$$

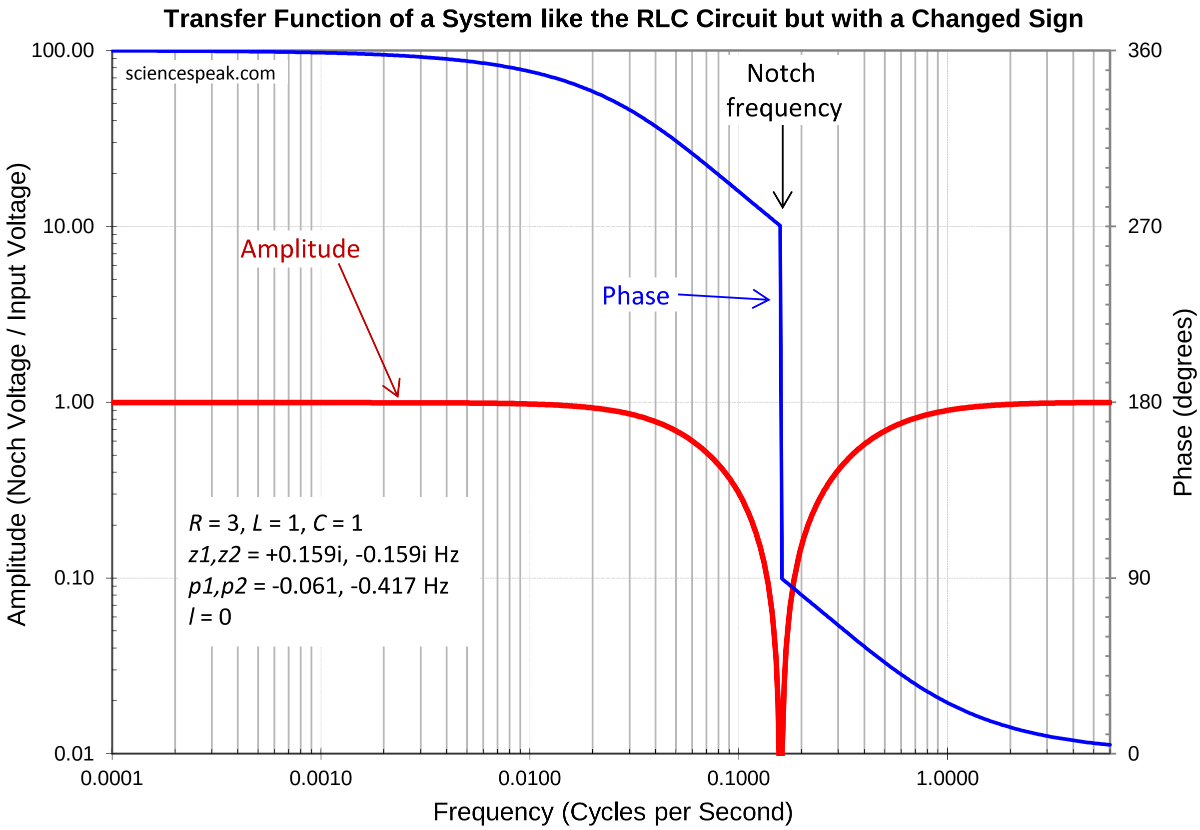


Figure 18: Transfer function of this system. Same as Fig. 16, except the sign of the phases is reversed.

10.5.4 Step Response from the Transfer Function

Substituting the mapping of Eq. (191) into the step response in Eq.s (132) and (133), the system's step response is

$$r_{\text{Notch},\mathbb{R}}(t) = \text{step}(t) - \frac{\alpha}{\beta}\text{step}(-t)\left[e^{(-\alpha-\beta)t} - e^{(-\alpha+\beta)t}\right], \tag{193}$$

which is non-causal, with the transients in the second term only existing when t is negative and dying away as t decreases. (k was not determined above, but it does not affect the step response because $\cos\theta_Z$ is zero. Thus the Fourier analysis gave only this one step response.)

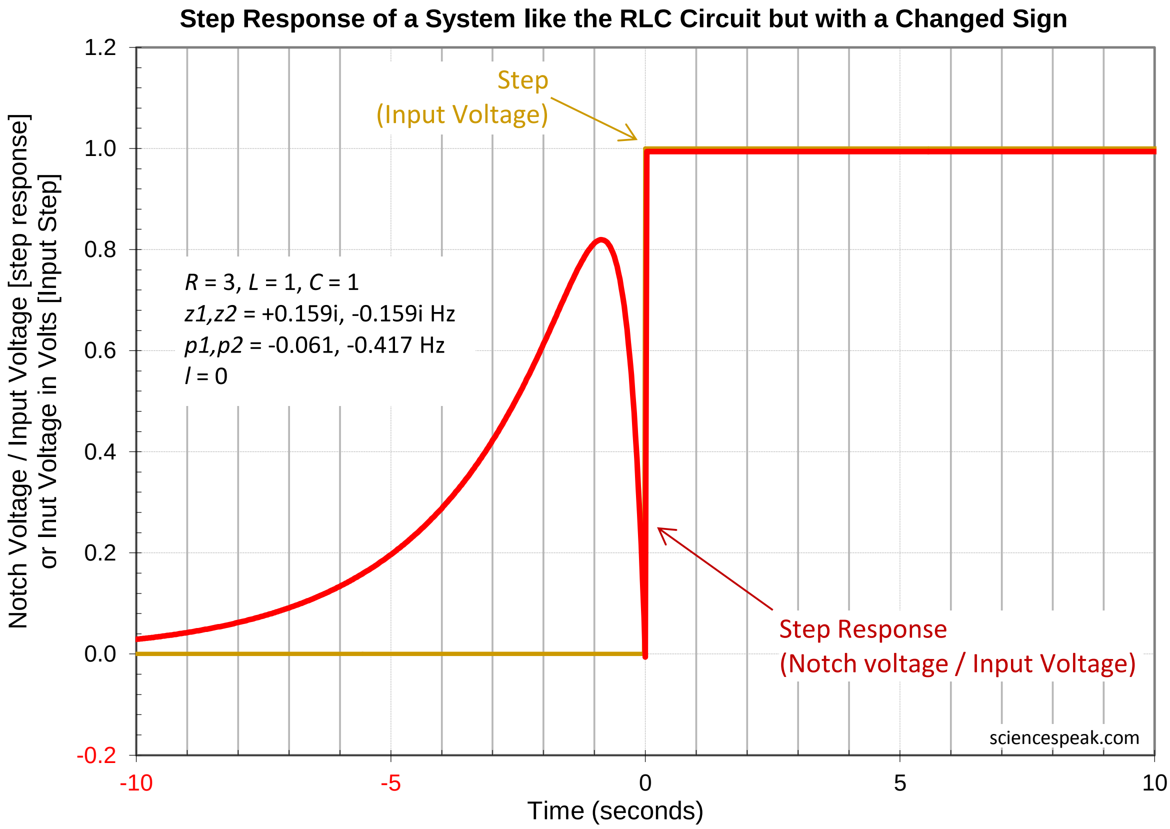


Figure 19: Step response of this system. It is non-causal. The transients "die out in negative time". Compare with Fig. 17, which is for a system with the same transfer function except the signs of the phases.

10.5.5 System Stability

Substituting the complex frequency s for $i2\pi f$ (because l is zero—see Eq. (83)), the Laplace-transform transfer function is

$$H_{\mathrm{N^*,s}}(s) = \frac{1+s^2LC}{1+s^2LC+sRC} = \frac{s^2+\dfrac{1}{LC}}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \tag{194}$$

the same as Eq. (177) (Laplace transforms are one-sided, ignoring everything before time zero, so different systems can have the same Laplace transform transfer function). Its poles, $\alpha \pm \beta$ (Eq.s (142) and (143)), are in the left hand side of the s-plane—so the system is stable. (Note that $2\pi d_1$ and $2\pi d_2$ are *defined* as the distances of the poles into the left hand of the *s*-plane in Eq. (97), and that they are positive.) If one substituted s for $-i2\pi f$ instead (as one would if l was one), then $H_{\mathrm{N^*,s}}$ would be the same except the sign of the s term in the denominator would be flipped, which would lead to positive poles—in which case the system would be unstable. The roots of the characteristic equation (Eq. (185)) are $-\alpha \pm \beta$, which are positive, so the complementary solution is

$$V_{\mathrm{N^*,C}}(t) = c_1 e^{(-\alpha+\beta)t} + c_2 e^{(-\alpha-\beta)t}, \tag{195}$$

so any transients grow larger with time—which would suggest that the system is unstable, except that the transients in the step response (Eq. (193)) only exist for times before zero (where they are always finite)! So this system appears to be stable but non-causal.

11 The Notch-Delay Solar Model

The notch-delay solar model is a specific arrangement of a notch filter, delay filter, and a low pass filter, and is itself a filter. It is organized into two parallel paths—a "direct" path consisting of just the low pass filter, and an "indirect" path consisting of the notch and delay filters and then the low pass filter. The model input is fed into each path; the model output is the sum of the outputs of the paths.

11.1 Step Response of the Indirect Path

The indirect path of the notch-delay filter consists of the notch filter, the delay filter, and the low pass filter, in a cascade. The transfer function of a cascade of LISs is the complex product of the transfer functions of the individual LISs, and because complex multiplication is commutative and associative, the individual LISs can be considered to be in any order. In order to use r_{Notch} and avoid calculating the impulse response of the notch filter or the inverse transform of the product of the individual transfer functions, we will analyze the indirect path as organized in Fig. 20.

Inputting a step function into the indirect path, the output of the notch filter is simply the step response of the notch filter r_{Notch}. The output of a LIS is the convolution of its input and its impulse response (Eq. (11)), so the output of the LPF is $r_{\text{Notch}} * h_{\text{LPF}}$ and the output of the entire indirect path is $r_{\text{Notch}} * h_{\text{LPF}} * h_{\text{Delay}}$.

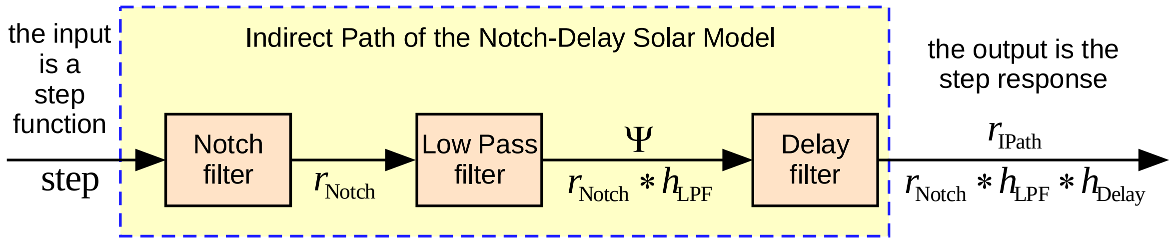


Figure 20: Finding the step response of the indirect path in the notch-delay filter.

11.1.1 Notch and Low Pass Filter

The step response of the combination of the notch and low pass filters is, by Eq. (71),

$$\begin{aligned}\Psi(t) &= \{r_{\text{Notch}} * h_{\text{LPF}}\}(t) \\ &= \int_{-\infty}^{\infty} r_{\text{Notch}}(u) h_{\text{LPF}}(t-u)\,du \\ &= 2\pi f_{\text{B}} w \int_{-\infty}^{\infty} r_{\text{Notch}}(u) \exp[-2\pi f_{\text{B}}(t-u)]\, \text{step}(t-u)\,du \\ &= 2\pi f_{\text{B}} w \exp(-2\pi f_{\text{B}} t) \int_{-\infty}^{t} r_{\text{Notch}}(u) \exp(2\pi f_{\text{B}} u)\,du \end{aligned} \tag{196}$$

11.1.2 Notch with Complex Poles and Low Pass Filter

With complex poles, r_{Notch} becomes the $r_{\text{Notch},\mathbb{C}}$ of Eq. (116). The character of $r_{\text{Notch},\mathbb{C}}$ is quite different depending on whether l is zero or one, so we treat them separately. The results here have been checked against numerical integrations of the product of Eq.s (94), (66), and (73) in Eq. (59).

If the notch filter has complex poles and l is zero, then $r_{\text{Notch},\mathbb{C}}$ is given by Eq. (118). If $t \le 0$ then $\Psi(t)$ is

$$\begin{aligned}\Psi_{\mathbb{C},0}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{-\infty}^{t} r_{\text{Notch},\mathbb{C},0}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{-\infty}^{t} \exp\left(\omega_{\mathrm{C}} u\right) \begin{bmatrix} -A_{\mathbb{C}} \cos\left(\omega_{\mathrm{S}} u\right) \\ +B_{\mathbb{C}} \sin\left(\omega_{\mathrm{S}} u\right) \end{bmatrix} \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \begin{bmatrix} -A_{\mathbb{C}} \int_{-\infty}^{t} \exp\left(\sigma u\right) \cos\left(\omega_{\mathrm{S}} u\right) du \\ +B_{\mathbb{C}} \int_{-\infty}^{t} \exp\left(\sigma u\right) \sin\left(\omega_{\mathrm{S}} u\right) du \end{bmatrix}\end{aligned} \tag{197}$$

where

$$\sigma = \omega_{\mathrm{C}} + 2\pi f_{\mathrm{B}}, \tag{198}$$

so by [Gradshteyn & Ryzhik, 1980, pp. 195, 2.662#2,#1]

$$\begin{aligned}\Psi_{\mathbb{C},0}(t) &= \frac{2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)}{\sigma^2 + \omega_{\mathrm{S}}^2} \left\{ \begin{matrix} -A_{\mathbb{C}} \exp\left(\sigma u\right)\left[\sigma \cos\left(\omega_{\mathrm{S}} u\right) + \omega_{\mathrm{S}} \sin\left(\omega_{\mathrm{S}} u\right)\right] \\ +B_{\mathbb{C}} \exp\left(\sigma u\right)\left[\sigma \sin\left(\omega_{\mathrm{S}} u\right) - \omega_{\mathrm{S}} \cos\left(\omega_{\mathrm{S}} u\right)\right] \end{matrix} \right\}_{u=-\infty}^{t} \\ &= \frac{2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)}{\sigma^2 + \omega_{\mathrm{S}}^2} \left\{ \begin{matrix} -A_{\mathbb{C}} \exp\left(\sigma t\right)\left[\sigma \cos\left(\omega_{\mathrm{S}} t\right) + \omega_{\mathrm{S}} \sin\left(\omega_{\mathrm{S}} t\right)\right] \\ +B_{\mathbb{C}} \exp\left(\sigma t\right)\left[\sigma \sin\left(\omega_{\mathrm{S}} t\right) - \omega_{\mathrm{S}} \cos\left(\omega_{\mathrm{S}} t\right)\right] \end{matrix} \right\} \\ &= -2\pi f_{\mathrm{B}} w \exp\left(\omega_{\mathrm{C}} t\right) \frac{\left(A_{\mathbb{C}}\sigma + B_{\mathbb{C}}\omega_{\mathrm{S}}\right)\cos\left(\omega_{\mathrm{S}} t\right) + \left(A_{\mathbb{C}}\omega_{\mathrm{S}} - B_{\mathbb{C}}\sigma\right)\sin\left(\omega_{\mathrm{S}} t\right)}{\sigma^2 + \omega_{\mathrm{S}}^2}.\end{aligned} \tag{199}$$

and

$$\begin{aligned}\Psi_{\mathbb{C},0}(0) &= 2\pi f_{\mathrm{B}} w \int_{-\infty}^{0} r_{\text{Notch},\mathbb{C},0}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ &= -2\pi f_{\mathrm{B}} w \frac{\left(A_{\mathbb{C}}\sigma + B_{\mathbb{C}}\omega_{\mathrm{S}}\right)}{\sigma^2 + \omega_{\mathrm{S}}^2}.\end{aligned} \tag{200}$$

For $t > 0$, $\Psi(t)$ is

$$\begin{aligned}\Psi_{\mathbb{C},0}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \begin{bmatrix} \int_{-\infty}^{0} r_{\text{Notch},\mathbb{C},0}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ +\int_{0}^{t} r_{\text{Notch},\mathbb{C},0}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du \end{bmatrix} \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \left\{ \frac{\Psi_{\mathbb{C}}(0)}{2\pi f_{\mathrm{B}} k} + \int_{0}^{t} \rho^2 \exp\left(2\pi f_{\mathrm{B}} u\right) du \right\} \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \left\{ \frac{\Psi_{\mathbb{C}}(0)}{2\pi f_{\mathrm{B}} k} + \rho^2 \frac{\exp\left(2\pi f_{\mathrm{B}} t\right) - 1}{2\pi f_{\mathrm{B}}} \right\} \\ &= \Psi_{\mathbb{C}}(0) \exp\left(-2\pi f_{\mathrm{B}} t\right) + w\rho^2 \left[1 - \exp\left(-2\pi f_{\mathrm{B}} t\right)\right] \\ &= w\rho^2 - \left[w\rho^2 - \Psi_{\mathbb{C}}(0)\right] \exp\left(-2\pi f_{\mathrm{B}} t\right).\end{aligned} \tag{201}$$

Hence the step response of the notch-LPF cascade, when the notch has complex poles and l is zero, is

$$\Psi_{\mathbb{C},0}(t) = w\begin{cases} -2\pi f_{\mathrm{B}} \exp\left(\omega_{\mathrm{C}} t\right) \dfrac{\left(A_{\mathbb{C}}\sigma + B_{\mathbb{C}}\omega_{\mathrm{S}}\right)\cos\left(\omega_{\mathrm{S}} t\right) + \left(A_{\mathbb{C}}\omega_{\mathrm{S}} - B_{\mathbb{C}}\sigma\right)\sin\left(\omega_{\mathrm{S}} t\right)}{\sigma^2 + \omega_{\mathrm{S}}^2} & \text{if } t \le 0 \\ \rho^2 - \left[\rho^2 - 2\pi f_{\mathrm{B}} \dfrac{\left(A_{\mathbb{C}}\sigma + B_{\mathbb{C}}\omega_{\mathrm{S}}\right)}{\sigma^2 + \omega_{\mathrm{S}}^2}\right] \exp\left(-2\pi f_{\mathrm{B}} t\right) & \text{if } t > 0. \end{cases} \tag{202}$$

If the notch filter has complex poles and l is one instead, then $r_{\mathrm{Notch},\mathbb{C}}$ is given by Eq. (119). If $t \le 0$ then $\Psi(t)$ is, by Eq. (196),

$$\Psi_{\mathbb{C},1}(t) = 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{-\infty}^{t} r_{\mathrm{Notch},\mathbb{C},1}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du = 0\,. \tag{203}$$

For $t > 0$, $\Psi(t)$ is

$$\begin{aligned} \Psi_{\mathbb{C},1}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{-\infty}^{t} r_{\mathrm{Notch},\mathbb{C},1}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{0}^{t} \left\{ \rho^2 + \exp\left(-\omega_{\mathrm{C}} u\right) \begin{bmatrix} +A_{\mathbb{C}} \cos\left(\omega_{\mathrm{S}} u\right) \\ +B_{\mathbb{C}} \sin\left(\omega_{\mathrm{S}} u\right) \end{bmatrix} \right\} \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \begin{Bmatrix} +\rho^2 \int_0^t \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ +A_{\mathbb{C}} \int_0^t \exp\left(\chi u\right) \cos\left(\omega_{\mathrm{S}} u\right) du \\ +B_{\mathbb{C}} \int_0^t \exp\left(\chi u\right) \sin\left(\omega_{\mathrm{S}} u\right) du \end{Bmatrix} \end{aligned} \tag{204}$$

where

$$\chi = -\omega_{\mathrm{C}} + 2\pi f_{\mathrm{B}}\,, \tag{205}$$

so by [Gradshteyn & Ryzhik, 1980, pp. 195, 2.662#2,#1]

$$
\begin{aligned}
\Psi_{\mathbb{C},1}(t) &= w\rho^2 \exp\left(-2\pi f_{\mathrm{B}} t\right) \exp\left(2\pi f_{\mathrm{B}} u\right)\Big|_{u=0}^{t} \\
&\quad + \frac{2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)}{\chi^2 + \omega_{\mathrm{S}}^2} \left\{ \begin{array}{l} +A_{\mathbb{C}} \exp\left(\chi u\right)\left[\chi \cos\left(\omega_{\mathrm{S}} u\right) + \omega_{\mathrm{S}} \sin\left(\omega_{\mathrm{S}} u\right)\right] \\ +B_{\mathbb{C}} \exp\left(\chi u\right)\left[\chi \sin\left(\omega_{\mathrm{S}} u\right) - \omega_{\mathrm{S}} \cos\left(\omega_{\mathrm{S}} u\right)\right] \end{array} \right\}_{u=0}^{t} \\
&= w\rho^2 \exp\left(-2\pi f_{\mathrm{B}} t\right)\left[\exp\left(2\pi f_{\mathrm{B}} t\right) - 1\right] \\
&\quad + \frac{2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)}{\chi^2 + \omega_{\mathrm{S}}^2} \left\{ \begin{array}{l} +A_{\mathbb{C}} \exp\left(\chi t\right)\left[\chi \cos\left(\omega_{\mathrm{S}} t\right) + \omega_{\mathrm{S}} \sin\left(\omega_{\mathrm{S}} t\right)\right] \\ +B_{\mathbb{C}} \exp\left(\chi t\right)\left[\chi \sin\left(\omega_{\mathrm{S}} t\right) - \omega_{\mathrm{S}} \cos\left(\omega_{\mathrm{S}} t\right)\right] \\ -A_{\mathbb{C}}\chi + B_{\mathbb{C}}\omega_{\mathrm{S}} \end{array} \right\} \\
&= w\rho^2 \left[1 - \exp\left(-2\pi f_{\mathrm{B}} t\right)\right] \\
&\quad + \frac{2\pi f_{\mathrm{B}} w}{\chi^2 + \omega_{\mathrm{S}}^2} \left\{ \begin{array}{l} +A_{\mathbb{C}} \exp\left(-\omega_{\mathrm{C}} t\right)\left[\chi \cos\left(\omega_{\mathrm{S}} t\right) + \omega_{\mathrm{S}} \sin\left(\omega_{\mathrm{S}} t\right)\right] \\ +B_{\mathbb{C}} \exp\left(-\omega_{\mathrm{C}} t\right)\left[\chi \sin\left(\omega_{\mathrm{S}} t\right) - \omega_{\mathrm{S}} \cos\left(\omega_{\mathrm{S}} t\right)\right] \\ -\exp\left(-2\pi f_{\mathrm{B}} t\right)\left[A_{\mathbb{C}}\chi - B_{\mathbb{C}}\omega_{\mathrm{S}}\right] \end{array} \right\}.
\end{aligned} \tag{206}
$$

Hence the step response of the notch-LPF cascade, when the notch has complex poles and *l* is one, is

$$
\Psi_{\mathbb{C},1}(t) = w \begin{cases} 0 & \text{if } t \le 0 \\ \rho^2 \left[1 - \exp\left(-2\pi f_{\mathrm{B}} t\right)\right] & \\ \quad - \dfrac{2\pi f_{\mathrm{B}}}{\chi^2 + \omega_{\mathrm{S}}^2} \exp\left(-2\pi f_{\mathrm{B}} t\right)\left[A_{\mathbb{C}}\chi - B_{\mathbb{C}}\omega_{\mathrm{S}}\right] & \\ \quad + \dfrac{2\pi f_{\mathrm{B}}}{\chi^2 + \omega_{\mathrm{S}}^2} \exp\left(-\omega_{\mathrm{C}} t\right) \left\{ \begin{array}{l} +\left[A_{\mathbb{C}}\chi - B_{\mathbb{C}}\omega_{\mathrm{S}}\right]\cos\left(\omega_{\mathrm{S}} t\right) \\ +\left[A_{\mathbb{C}}\omega_{\mathrm{S}} + B_{\mathbb{C}}\chi\right]\sin\left(\omega_{\mathrm{S}} t\right) \end{array} \right\} & \text{if } t > 0. \end{cases} \tag{207}
$$

Our example is continued in Fig. 21.

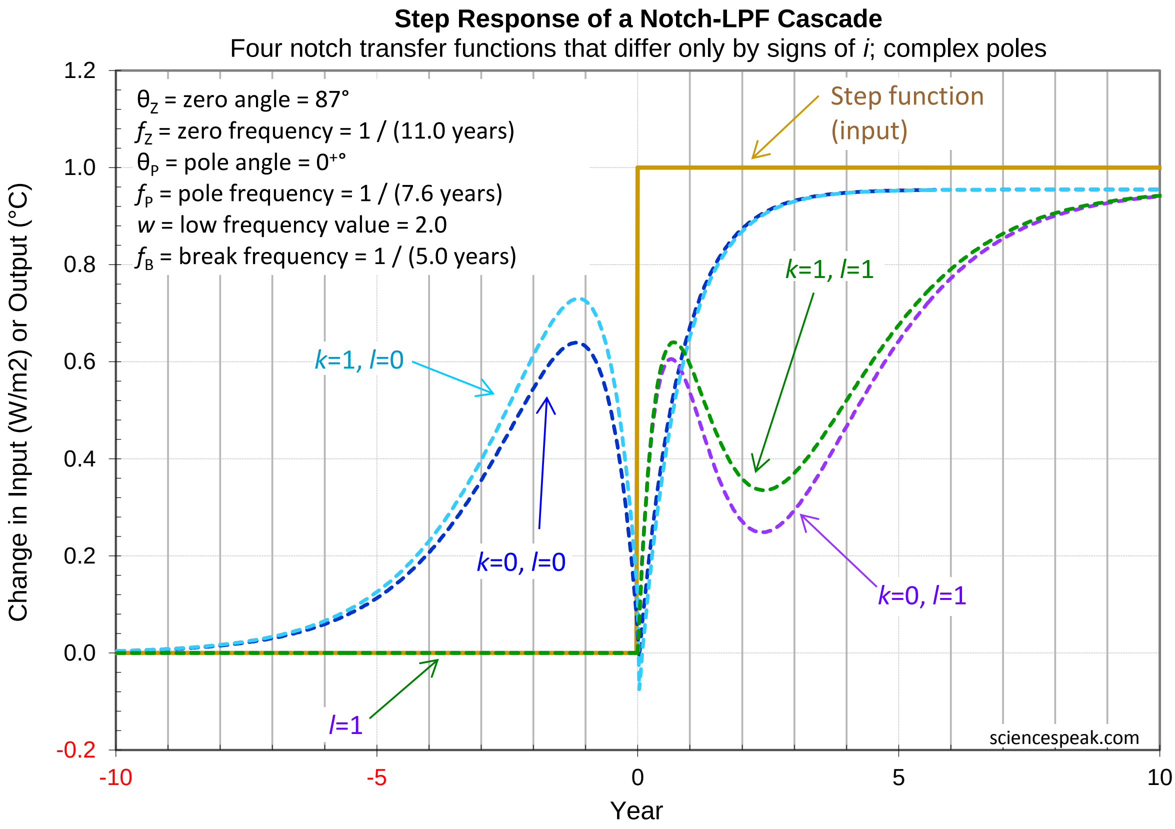


Figure 21: The step response of a notch filter in cascade with a low pass filter. These are the same filters as in Fig. 13 (showing the four notch transfer functions that differ only by signs of *i*) and Fig. 6. Notice how the low pass filter smooths the notch step response, especially the prominent sharp corners at the time origin.

11.1.3 Notch with Real Poles and Low Pass Filter

With real poles, r_{Notch} becomes the $r_{\text{Notch},\mathbb{R}}$ of Eq. (132). The character of $r_{\text{Notch},\mathbb{R}}$ is quite different depending on whether l is zero or one, so we treat them separately. The results here have been checked against numerical integrations of the product of Eq.s (99), (66), and (73) in Eq. (59).

If the notch filter has real poles and l is zero, then $r_{\text{Notch},\mathbb{R}}$ is given by Eq. (134) . If $t \le 0$ then $\Psi(t)$ is, by Eq. (196),

$$\begin{aligned}\Psi_{\mathbb{R},0}(t) &= 2\pi f_{\text{B}} w \exp\left(-2\pi f_{\text{B}} t\right) \int_{-\infty}^{t} r_{\text{Notch},\mathbb{R},0}(u) \exp\left(2\pi f_{\text{B}} u\right) du \\ &= 2\pi f_{\text{B}} w \exp\left(-2\pi f_{\text{B}} t\right) \int_{-\infty}^{t} \begin{bmatrix} -A_{\mathbb{R}} \exp\left(2\pi d_1 u\right) \\ +B_{\mathbb{R}} \exp\left(2\pi d_2 u\right) \end{bmatrix} \exp\left(2\pi f_{\text{B}} u\right) du \\ &= 2\pi f_{\text{B}} w \exp\left(-2\pi f_{\text{B}} t\right) \begin{bmatrix} -A_{\mathbb{R}} \int_{-\infty}^{t} \exp\left(\kappa_1 u\right) du \\ +B_{\mathbb{R}} \int_{-\infty}^{t} \exp\left(\kappa_2 u\right) du \end{bmatrix}\end{aligned} \tag{208}$$

where

$$\kappa_1 = 2\pi\left(f_{\text{B}} + d_1\right) \quad \text{and} \quad \kappa_2 = 2\pi\left(f_{\text{B}} + d_2\right), \tag{209}$$

so

$$\begin{aligned}\Psi_{\mathbb{R},0}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)\left\{-\frac{A_{\mathbb{R}}}{\kappa_1}\exp\left(\kappa_1 u\right)+\frac{B_{\mathbb{R}}}{\kappa_2}\exp\left(\kappa_2 u\right)\right\}_{u=-\infty}^{t} \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)\left\{-\frac{A_{\mathbb{R}}}{\kappa_1}\exp\left(\kappa_1 t\right)+\frac{B_{\mathbb{R}}}{\kappa_2}\exp\left(\kappa_2 t\right)\right\} \\ &= 2\pi f_{\mathrm{B}} w \left[-\frac{A_{\mathbb{R}}}{\kappa_1}\exp\left(2\pi d_1 t\right)+\frac{B_{\mathbb{R}}}{\kappa_2}\exp\left(2\pi d_2 t\right)\right]\end{aligned} \tag{210}$$

and

$$\begin{aligned}\Psi_{\mathbb{R},0}(0) &= 2\pi f_{\mathrm{B}} w \int_{-\infty}^{0} r_{\mathrm{Notch},\mathbb{R},0}(u)\exp\left(2\pi f_{\mathrm{B}} u\right)du \\ &= 2\pi f_{\mathrm{B}} w\left(-\frac{A_{\mathbb{R}}}{\kappa_1}+\frac{B_{\mathbb{R}}}{\kappa_2}\right).\end{aligned} \tag{211}$$

For $t > 0$, $\Psi(t)$ is

$$\begin{aligned}\Psi_{\mathbb{R},0}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)\begin{bmatrix}+\int_{-\infty}^{0} r_{\mathrm{Notch},\mathbb{R},0}(u)\exp\left(2\pi f_{\mathrm{B}} u\right)du \\ +\int_{0}^{t} r_{\mathrm{Notch},\mathbb{R},0}(u)\exp\left(2\pi f_{\mathrm{B}} u\right)du\end{bmatrix} \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)\left\{\frac{\Psi_{\mathbb{R},0}(0)}{2\pi f_{\mathrm{B}} w}+\int_0^t \frac{f_{\mathrm{Z}}^2}{d_1 d_2}\exp\left(2\pi f_{\mathrm{B}} u\right)du\right\} \\ &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)\left\{\frac{\Psi_{\mathbb{R}}(0)}{2\pi f_{\mathrm{B}} w}+\frac{f_{\mathrm{Z}}^2}{d_1 d_2}\frac{\exp\left(2\pi f_{\mathrm{B}} t\right)-1}{2\pi f_{\mathrm{B}}}\right\} \\ &= \Psi_{\mathbb{R}}(0)\exp\left(-2\pi f_{\mathrm{B}} t\right)+w\frac{f_{\mathrm{Z}}^2}{d_1 d_2}\left[1-\exp\left(-2\pi f_{\mathrm{B}} t\right)\right] \\ &= w\frac{f_{\mathrm{Z}}^2}{d_1 d_2}+\left[\Psi_{\mathbb{R}}(0)-w\frac{f_{\mathrm{Z}}^2}{d_1 d_2}\right]\exp\left(-2\pi f_{\mathrm{B}} t\right).\end{aligned} \tag{212}$$

Hence the step response of the notch-LPF cascade, when the notch has real poles and *l* is zero, is

$$\Psi_{\mathbb{R},0}(t) = w\begin{cases} f_{\mathrm{B}}\left[-\dfrac{A_{\mathbb{R}}}{f_{\mathrm{B}}+d_1}\exp\left(2\pi d_1 t\right)+\dfrac{B_{\mathbb{R}}}{f_{\mathrm{B}}+d_2}\exp\left(2\pi d_2 t\right)\right] & \text{if } t \le 0 \\ \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2}+\left[f_{\mathrm{B}}\left(-\dfrac{A_{\mathbb{R}}}{f_{\mathrm{B}}+d_1}+\dfrac{B_{\mathbb{R}}}{f_{\mathrm{B}}+d_2}\right)-\dfrac{f_{\mathrm{Z}}^2}{d_1 d_2}\right]\exp\left(-2\pi f_{\mathrm{B}} t\right) & \text{if } t > 0.\end{cases} \tag{213}$$

If the notch filter has real poles and *l* is one instead, then $r_{\mathrm{Notch},\mathbb{R}}$ is given by Eq. (135). If $t \le 0$ then $\Psi(t)$ is, by Eq. (196),

$$\Psi_{\mathbb{R},1}(t) = 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right)\int_{-\infty}^{t} r_{\mathrm{Notch},\mathbb{R},1}(u)\exp\left(2\pi f_{\mathrm{B}} u\right)du = 0. \tag{214}$$

For $t > 0$, $\Psi(t)$ is

$$\begin{aligned}
\Psi_{\mathbb{R},1}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{-\infty}^{t} r_{\mathrm{Notch},\mathbb{R},1}(u) \exp\left(2\pi f_{\mathrm{B}} u\right) du \\
&= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \int_{0}^{t} \begin{bmatrix} \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} + A_{\mathbb{R}} \exp\left(-2\pi d_1 u\right) \\ - B_{\mathbb{R}} \exp\left(-2\pi d_2 u\right) \end{bmatrix} \exp\left(2\pi f_{\mathrm{B}} u\right) du \\
&= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \begin{bmatrix} + \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} \displaystyle\int_{0}^{t} \exp\left(2\pi f_{\mathrm{B}} u\right) du \\ + A_{\mathbb{R}} \displaystyle\int_{0}^{t} \exp\left(\upsilon_1 u\right) du \\ - B_{\mathbb{R}} \displaystyle\int_{0}^{t} \exp\left(\upsilon_2 u\right) du \end{bmatrix}
\end{aligned} \tag{215}$$

where

$$\upsilon_1 = 2\pi\left(f_{\mathrm{B}} - d_1\right) \quad \text{and} \quad \upsilon_2 = 2\pi\left(f_{\mathrm{B}} - d_2\right), \tag{216}$$

so

$$\begin{aligned}
\Psi_{\mathbb{R},1}(t) &= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \left[\frac{f_{\mathrm{Z}}^2}{d_1 d_2} \frac{\exp\left(2\pi f_{\mathrm{B}} u\right)}{2\pi f_{\mathrm{B}}} + \frac{A_{\mathbb{R}}}{\upsilon_1} \exp\left(\upsilon_1 u\right) - \frac{B_{\mathbb{R}}}{\upsilon_2} \exp\left(\upsilon_2 u\right) \right]_{u=0}^{t} \\
&= 2\pi f_{\mathrm{B}} w \exp\left(-2\pi f_{\mathrm{B}} t\right) \begin{bmatrix} \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} \dfrac{\exp\left(2\pi f_{\mathrm{B}} t\right)}{2\pi f_{\mathrm{B}}} + \dfrac{A_{\mathbb{R}}}{\upsilon_1} \exp\left(\upsilon_1 t\right) - \dfrac{B_{\mathbb{R}}}{\upsilon_2} \exp\left(\upsilon_2 t\right) \\ - \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} \dfrac{1}{2\pi f_{\mathrm{B}}} - \dfrac{A_{\mathbb{R}}}{\upsilon_1} + \dfrac{B_{\mathbb{R}}}{\upsilon_2} \end{bmatrix} \\
&= f_{\mathrm{B}} w \begin{bmatrix} \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} \dfrac{1}{f_{\mathrm{B}}} + \dfrac{A_{\mathbb{R}}}{f_{\mathrm{B}} - d_1} \exp\left(-2\pi d_1 t\right) - \dfrac{B_{\mathbb{R}}}{f_{\mathrm{B}} - d_2} \exp\left(-2\pi d_2 t\right) \\ - \left(\dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} \dfrac{1}{f_{\mathrm{B}}} + \dfrac{A_{\mathbb{R}}}{f_{\mathrm{B}} - d_1} - \dfrac{B_{\mathbb{R}}}{f_{\mathrm{B}} - d_2} \right) \exp\left(-2\pi f_{\mathrm{B}} t\right) \end{bmatrix}.
\end{aligned} \tag{217}$$

Hence the step response of the notch-LPF cascade, when the notch has real poles and *l* is one, is

$$\Psi_{\mathbb{R},1}(t) = w \begin{cases} 0 & \text{if } t \le 0 \\ \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} + f_{\mathrm{B}} \left[\dfrac{A_{\mathbb{R}}}{f_{\mathrm{B}} - d_1} \exp\left(-2\pi d_1 t\right) - \dfrac{B_{\mathbb{R}}}{f_{\mathrm{B}} - d_2} \exp\left(-2\pi d_2 t\right) \right] \\ \quad - \left[f_{\mathrm{B}} \left(\dfrac{A_{\mathbb{R}}}{f_{\mathrm{B}} - d_1} - \dfrac{B_{\mathbb{R}}}{f_{\mathrm{B}} - d_2} \right) + \dfrac{f_{\mathrm{Z}}^2}{d_1 d_2} \right] \exp\left(-2\pi f_{\mathrm{B}} t\right) & \text{if } t > 0. \end{cases} \tag{218}$$

11.1.4 Notch, Low Pass Filter, and Delay

Now let us add in the effect of the delay filter (see Fig. 20). The step response of the notch, low pass, and delay filters combined is, by Eq. (71),

$$
\begin{aligned}
r_{\mathrm{DPath}}(t) &= \left\{\left(r_{\mathrm{Notch}} * h_{\mathrm{LPF}}\right) * h_{\mathrm{Delay}}\right\}(t) \\
&= \int_{-\infty}^{\infty} \Psi(u) h_{\mathrm{Delay}}(t-u)\,du \\
&= \int_{-\infty}^{\infty} \Psi(u) \delta\left[(t-u)-d\right] du \\
&= \Psi(t-d),
\end{aligned}
\tag{219}
$$

which of course is the step response of the notch-LPF cascade delayed by d.

11.1.5 Notch with Complex Poles, Low Pass Filter, and Delay

The step response of the indirect path (i.e. notch, delay, and LPF), when the notch has complex poles and l is zero, is, by Eq.s (202) and (219),

$$
r_{\mathrm{IPath},\mathbb{C},0}(t) = w\begin{cases}
\dfrac{\left(A_{\mathbb{C}}\sigma + B_{\mathbb{C}}\omega_{\mathrm{S}}\right)\cos\left[\omega_{\mathrm{S}}(t-d)\right] + \left(A_{\mathbb{C}}\omega_{\mathrm{S}} - B_{\mathbb{C}}\sigma\right)\sin\left[\omega_{\mathrm{S}}(t-d)\right]}{\left\{-2\pi f_{\mathrm{B}} \exp\left[\omega_{\mathrm{C}}(t-d)\right]\right\}^{-1}\left(\sigma^2 + \omega_{\mathrm{S}}^2\right)} & \text{if } t \le d \\
\rho^2 - \left[\rho^2 - 2\pi f_{\mathrm{B}} \dfrac{\left(A_{\mathbb{C}}\sigma + B_{\mathbb{C}}\omega_{\mathrm{S}}\right)}{\sigma^2 + \omega_{\mathrm{S}}^2}\right] \exp\left[-2\pi f_{\mathrm{B}}(t-d)\right] & \text{if } t > d\,,
\end{cases}
\tag{220}
$$

where the parameters are defined by Eq.s (64), (72), (94), (117), and (198). To make it (almost) causal, d has to be positive—that is, the effect of the notch has to be delayed (not advanced). If l is one instead, the step response, by Eq. (207), is

$$
r_{\mathrm{IPath},\mathbb{C},1}(t) = w\begin{cases}
0 & \text{if } t \le d \\
\rho^2\left\{1 - \exp\left[-2\pi f_{\mathrm{B}}(t-d)\right]\right\} & \\
\quad - \dfrac{2\pi f_{\mathrm{B}}}{\chi^2 + \omega_{\mathrm{S}}^2} \exp\left[-2\pi f_{\mathrm{B}}(t-d)\right]\left[A_{\mathbb{C}}\chi - B_{\mathbb{C}}\omega_{\mathrm{S}}\right] & \\
\quad + \dfrac{2\pi f_{\mathrm{B}}}{\chi^2 + \omega_{\mathrm{S}}^2} \exp\left[-\omega_{\mathrm{C}}(t-d)\right]\begin{Bmatrix} +\left[A_{\mathbb{C}}\chi - B_{\mathbb{C}}\omega_{\mathrm{S}}\right]\cos\left[\omega_{\mathrm{S}}(t-d)\right] \\ +\left[A_{\mathbb{C}}\omega_{\mathrm{S}} + B_{\mathbb{C}}\chi\right]\sin\left[\omega_{\mathrm{S}}(t-d)\right] \end{Bmatrix} & \text{if } t > d\,,
\end{cases}
\tag{221}
$$

where the parameters are defined by Eq.s (64), (72), (94), (117), and (205). Our example continues in Fig. 22.

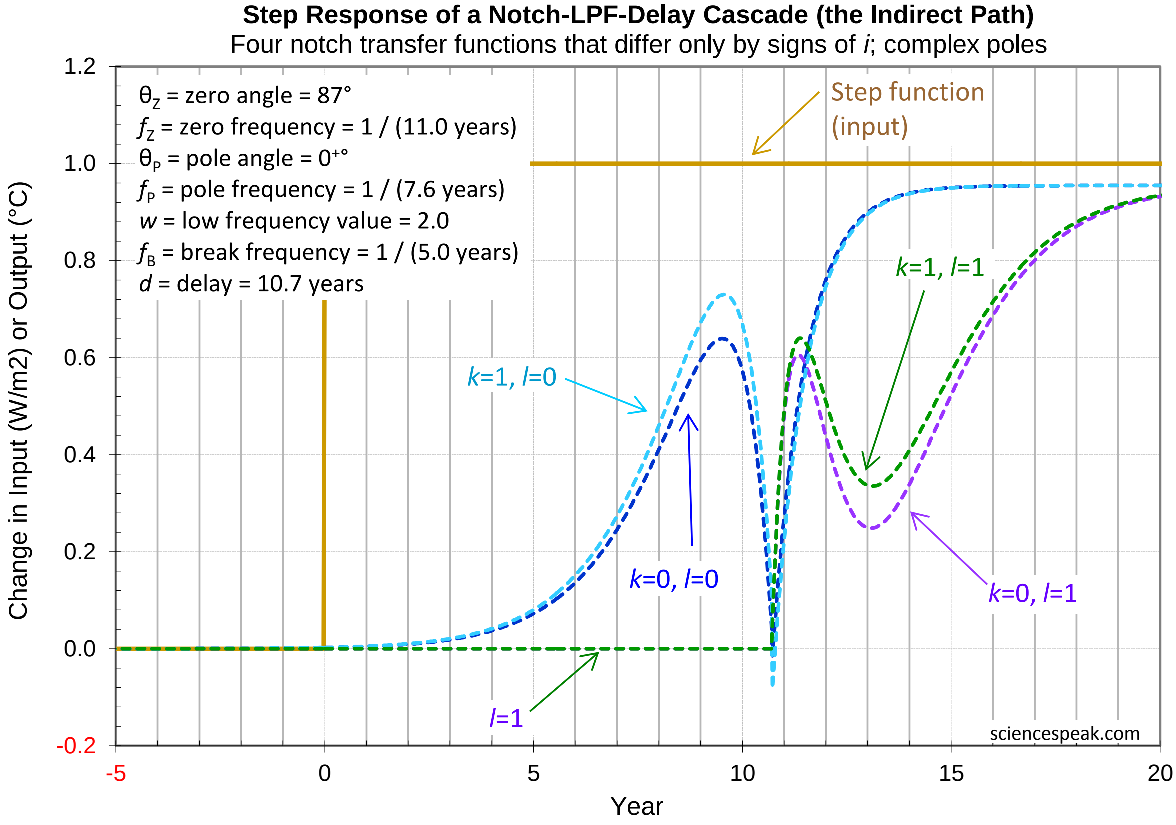


Figure 22: The step response of the indirect path in the Notch-Delay Solar Model, a notch filter in cascade with a low pass filter and a delay filter. As per Fig. 21, except now with a delay. The delay makes the step response causal.

11.1.6 Notch with Real Poles, Low Pass Filter, and Delay

The step response of the indirect path (i.e. notch, delay, and LPF), when the notch has real poles and l is zero, is, by Eq.s (213) and (219),

$$r_{\text{IPath},\mathbb{R},0}(t) = w \begin{cases} f_{\text{B}} \left\{ -\dfrac{A_{\mathbb{R}}}{f_{\text{B}}+d_1} \exp\left[2\pi d_1 (t-d)\right] + \dfrac{B_{\mathbb{R}}}{f_{\text{B}}+d_2} \exp\left[2\pi d_2 (t-d)\right] \right\} & \text{if } t \le d \\ \dfrac{f_{\text{Z}}^2}{d_1 d_2} + \left[f_{\text{B}} \left(-\dfrac{A_{\mathbb{R}}}{f_{\text{B}}+d_1} + \dfrac{B_{\mathbb{R}}}{f_{\text{B}}+d_2} \right) - \dfrac{f_{\text{Z}}^2}{d_1 d_2} \right] \exp\left[-2\pi f_{\text{B}} (t-d)\right] & \text{if } t > d, \end{cases} \tag{222}$$

where the parameters are defined by Eq.s (64), (72), (99), and (133). To make it (almost) causal, d has to be positive. If l is one instead, the step response is instead, by Eq. (218),

$$r_{\text{IPath},\mathbb{R},1}(t) = w \begin{cases} 0 & \text{if } t \le d \\ \dfrac{f_{\text{Z}}^2}{d_1 d_2} + f_{\text{B}} \left\{ \begin{array}{l} \dfrac{A_{\mathbb{R}}}{f_{\text{B}}-d_1} \exp\left[-2\pi d_1 (t-d)\right] \\ -\dfrac{B_{\mathbb{R}}}{f_{\text{B}}-d_2} \exp\left[-2\pi d_2 (t-d)\right] \end{array} \right\} & \\ \quad -\left[f_{\text{B}} \left(\dfrac{A_{\mathbb{R}}}{f_{\text{B}}-d_1} - \dfrac{B_{\mathbb{R}}}{f_{\text{B}}-d_2} \right) + \dfrac{f_{\text{Z}}^2}{d_1 d_2} \right] \exp\left[-2\pi f_{\text{B}} (t-d)\right] & \text{if } t > d, \end{cases} \tag{223}$$

where the parameters are defined by Eq.s (64), (72), (99), and (133).

11.2 Step Response of the direct Path

The direct path just has the low pass filter, so its step response is given by Eq. (70) and is shown in Fig. 6.

11.3 Step Response of the Notch-Delay Solar Model

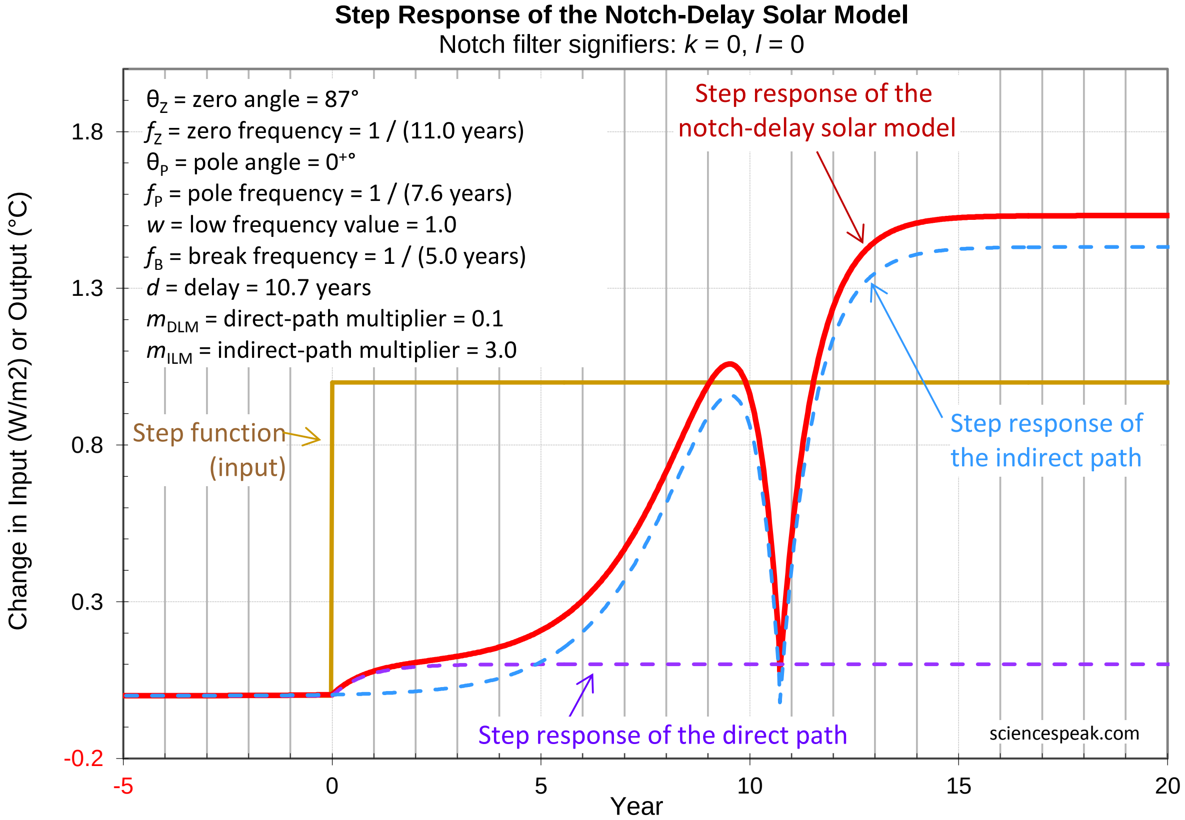
The notch-delay solar model is the sum of the direct and indirect paths, so when the notch filter has complex poles its step response is

$$r_{\mathrm{ND},\mathbb{C}}(t) = k\left[1 - \exp\left(-2\pi f_{\mathrm{B}} t\right)\right]\mathrm{step}(t) + r_{\mathrm{IPath},\mathbb{C},l}(t)\,, \quad l = 0,1 \tag{224}$$

(see Eq.s (220) and (221) for $r_{\mathrm{IPath},\mathbb{C},0}$ and $r_{\mathrm{IPath},\mathbb{C},1}$). See Fig. 23. When the notch filter has real poles, the step response of the notch-delay solar model is

$$r_{\mathrm{ND},\mathbb{R}}(t) = k\left[1 - \exp\left(-2\pi f_{\mathrm{B}} t\right)\right]\mathrm{step}(t) + r_{\mathrm{IPath},\mathbb{R},l}(t)\,, \quad l = 0,1 \tag{225}$$

(see Eq.s (222) and (223) for $r_{\mathrm{IPath},\mathbb{R},0}$ and $r_{\mathrm{IPath},\mathbb{R},1}$).



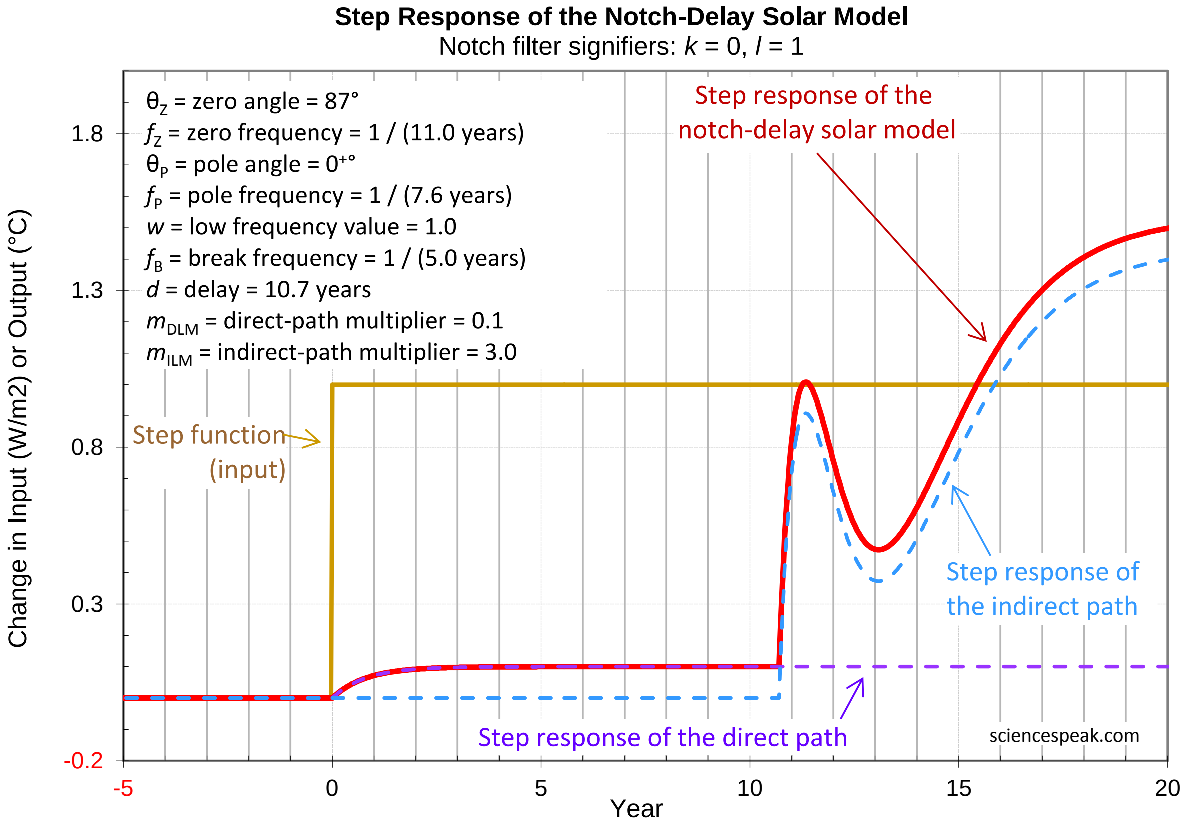


Figure 23: The step response of the notch-delay solar model is the sum of the step responses of its direct and indirect paths. Adds Fig.s 22 and 6 (after scaling by m_{DLM} and m_{ILM}).

11.4 Insight into the Notch-Delay Solar Model Filter

According to the notch-delay solar theory, the Sun is the main driver of changes in surface temperatures here on Earth. However the effect is *not* primarily due to the direct heating effect of changes in bulk sunlight, because they are much too small to have caused the global warming of the last few decades. Instead, it is changes in some aspect of the Sun (for example, perhaps the amount of extreme UV) that affect the Earth's albedo (for example, perhaps by ozone affecting the shape of jet streams and thus cloud formation, or by plankton manufacturing reflective aerosols). The hypothesis is that a solar force, at this stage unknown and called "force X", drives surface temperatures on Earth by modulating the Earth's albedo. A critical feature of force X is that changes in force X lag one sunspot cycle, or 11 years on average, behind corresponding changes in total solar irradiance (TSI).

("Force X"? Surely you must be joking, that's like something out of a cartoon. It is, but the cartoons got their inspiration from "x-rays"—whose discoverer, Wilhelm Röntgen, named them thus to signify an unknown type of radiation.)

We model this with a system whose input is TSI and whose output is surface temperature, both functions of time. To a first approximation, surface temperatures follow force X, which is proportional to the TSI delayed by roughly 11 years. That step response is shown in Fig. 24.

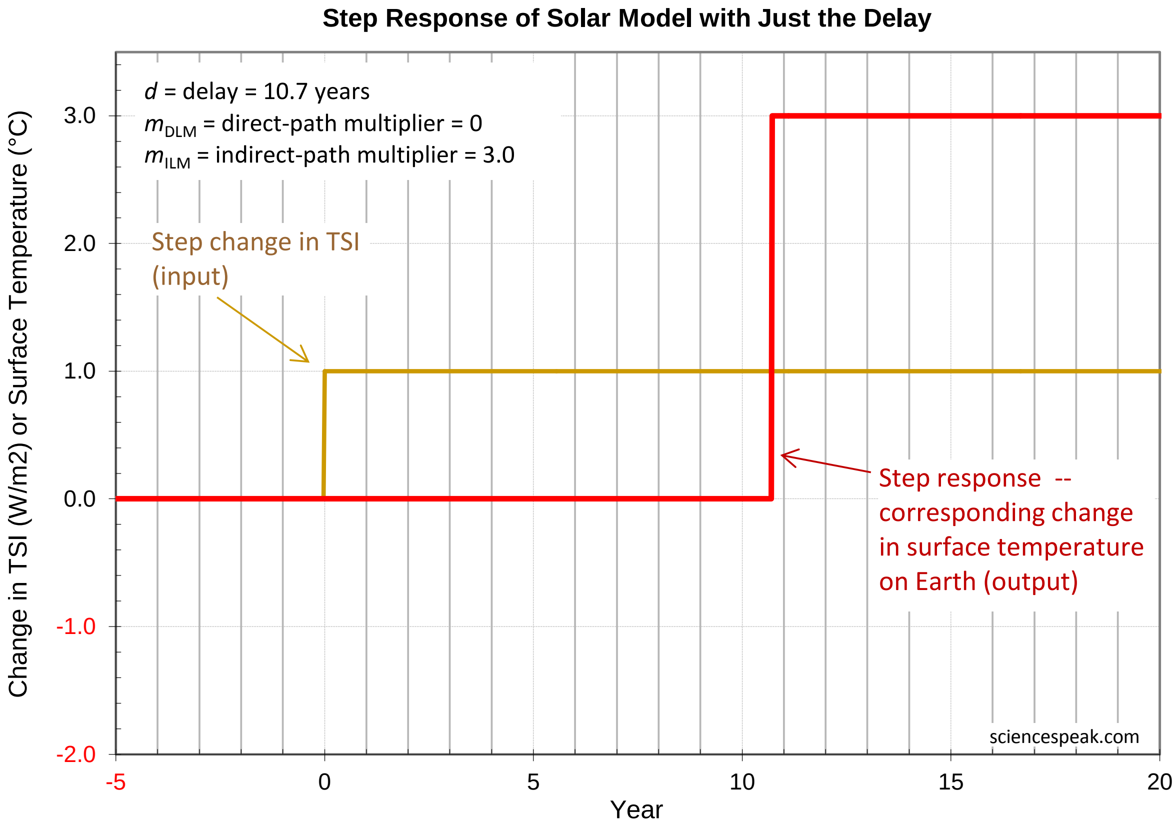


Figure 24: To a first approximation, the step response of the system whose input is TSI and whose output is surface temperature is a delay of about 10.7 years.

But force X is attenuated during the sunspot maxima, possibly dropping briefly to a minimum as the solar magnetic field reverses polarity. But during the sunspot maxima the TSI peaks slightly, which causes surface temperatures to rise by direct heating—the two effects largely cancel out, resulting in the observed notch at around 11 years in the transfer function from TSI to surface temperature. So, to a second approximation, force X is like the delayed TSI but with a notch filter applied to reduce force X during the TSI peaks that occur during the sunspot maxima.

Although a notch was observed in the empirical transfer function we don't know the phases of that transfer function, so the step response of the notch filter could be any of the eight basic types that fit the observed transfer function amplitude (see Fig. 11, 12, 13, 14, which show real and complex poles, with four combinations of signifiers *k* and *l*). But four of them are very similar to the causal step response, and four of them are like the non-causal step response, so we need only consider those two step responses. However the actual step response might be a mix of the two. See Fig. 25.

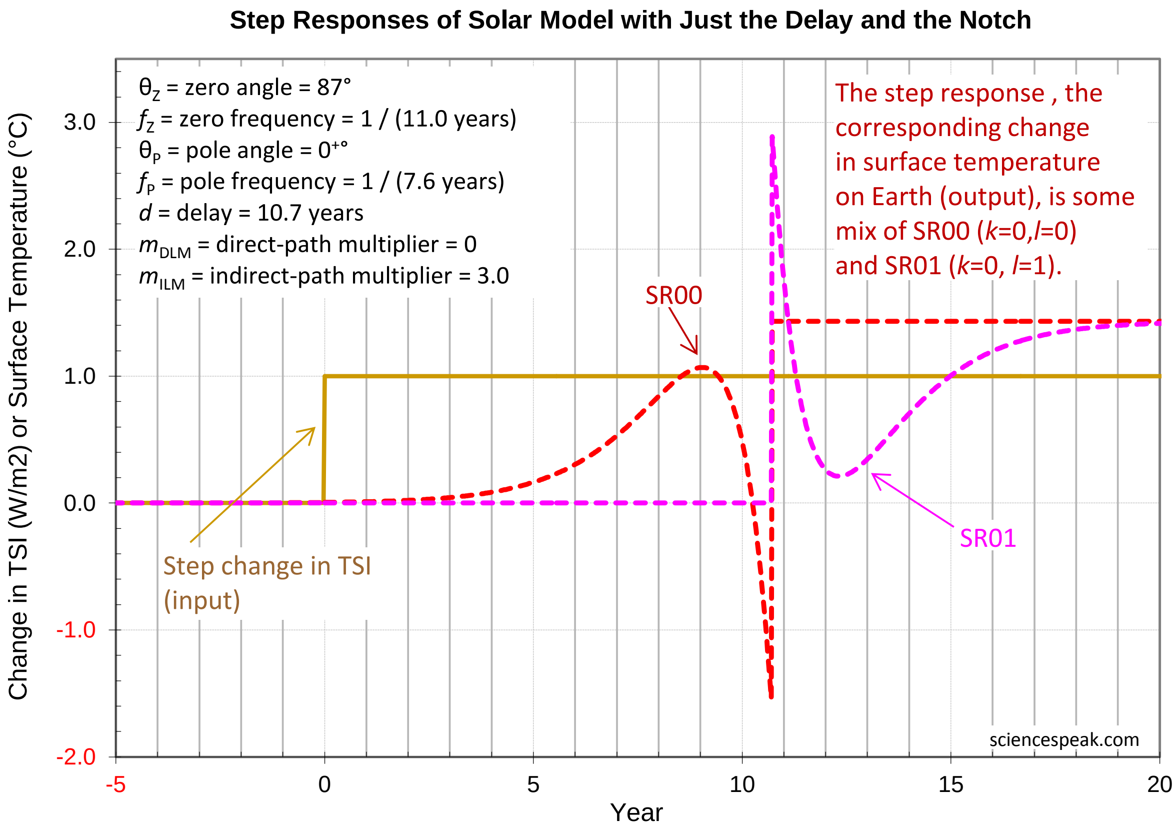


Figure 25: To a second approximation, the system is a delay and a notch. We are not yet sure which of the two step responses shown is appropriate.

But the Earth has a considerable thermal momentum: heat the planet by moving to a slightly higher level of extra sunlight or decreased albedo, and it takes a couple of years for the temperature to rise and level off to its new level. So, to a third approximation, we introduce a low pass filter to smooth out the temperature response of the Earth, as shown in Fig. 26.

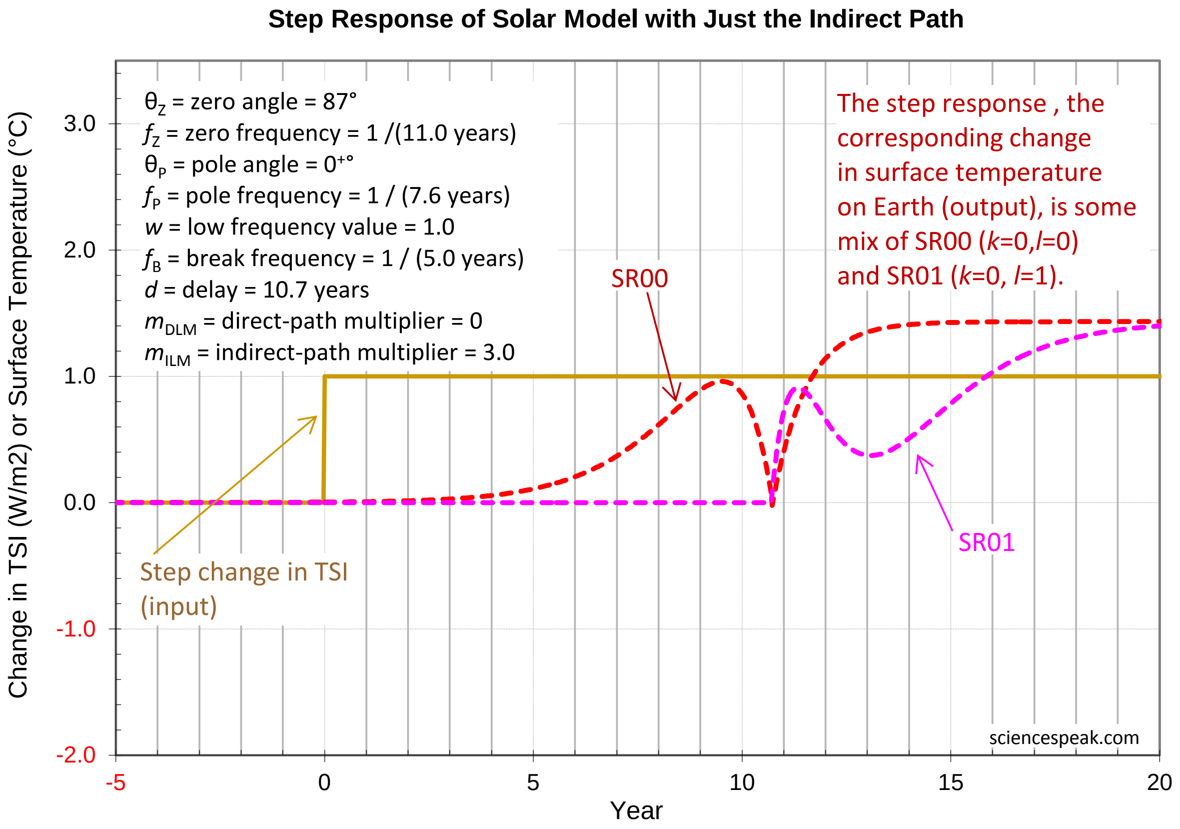


Figure 26: To a third approximation, the system is a delay, a notch, and a low pass filter.

Finally, the surface temperature is also changed by the direct heating effect of TSI changes. So we must introduce the direct path, in parallel with the notch and delay of the indirect path but sharing the same low pass filter. See Fig. 27.

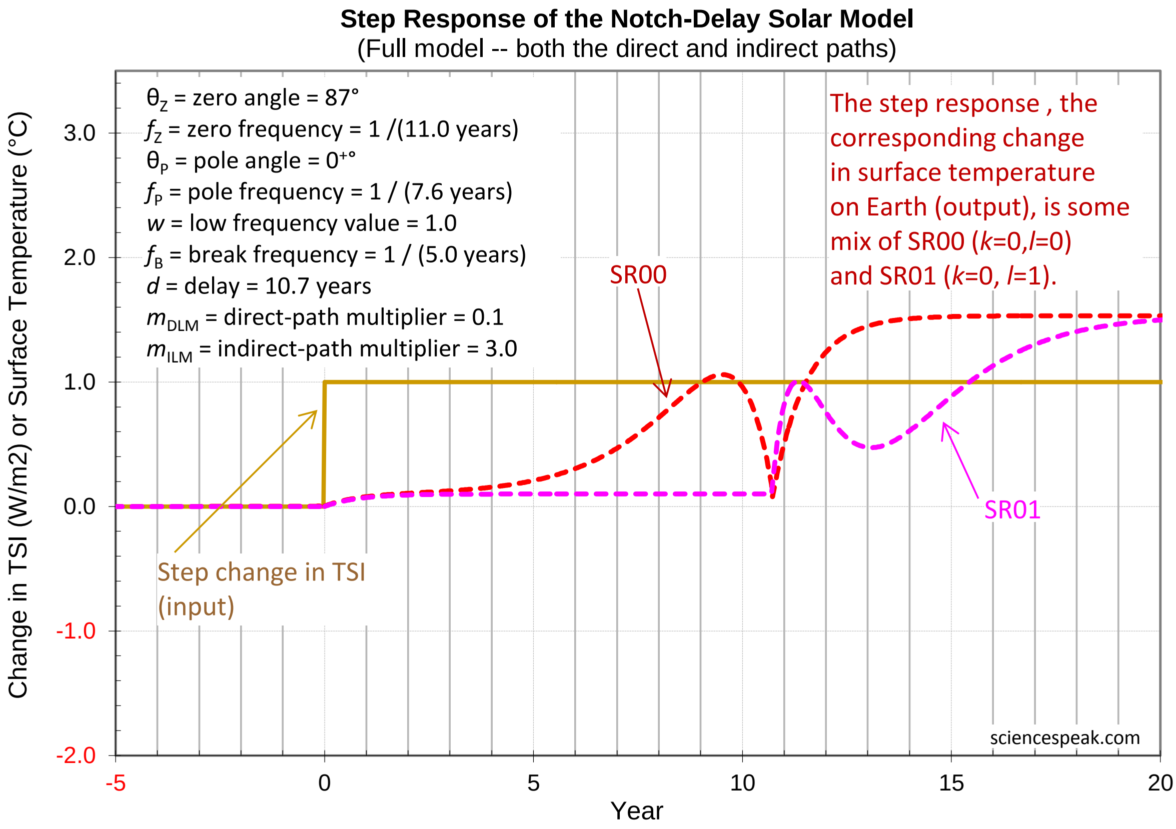


Figure 27: Full model, with a direct path in parallel with the indirect path. The effect of the direct path, for changes in direct heating of Earth by TSI, is of relatively minor.

11.5 Simple Approximation to the Notch-Delay Solar Model Filter

The step response of the ND solar model illustrated is approximated by a centered 11-year smoother, whose output at time t is simply the plain, unweighted average of the input over the interval $[t-5.5\,\text{years},\ t+5.5\,\text{years}]$, with the same delay, and scaled to match the final output of the ND solar model filter. The step responses of both the 11-year centered smoother and the notch filter (and even more so the notch-LPF cascade) crudely approximate the step response of the identity system. See Fig. 28.

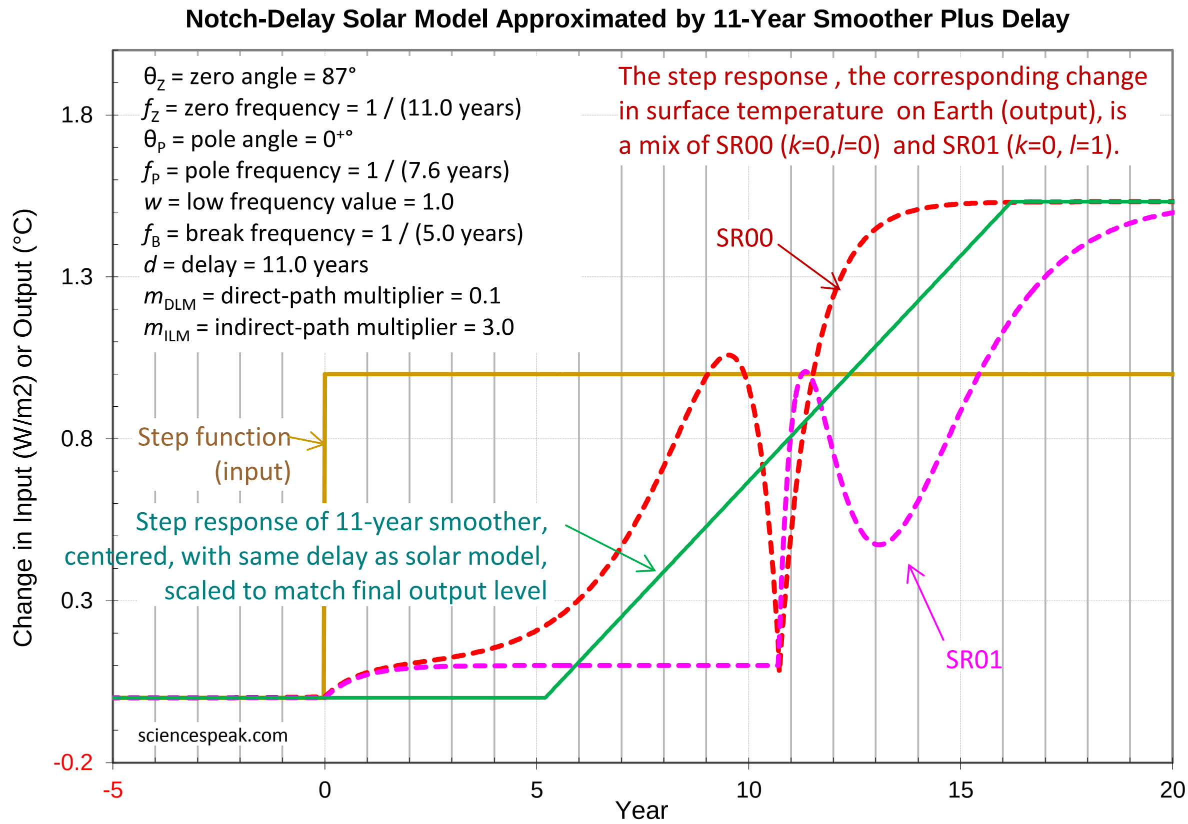


Figure 28: An 11-year smoother-with-delay is a crude approximation of the notch-delay solar model.

12 Conclusion

The context of this document is developing a theory of how the Sun affects the Earth's temperature, given that carbon dioxide has only a minor effect.

The sunspot record from 1610 is about all we have measured about the Sun, on a timescale of more than a few decades. Those sunspot numbers have been converted to estimates of total solar irradiance (TSI), by comparing TSI and sunspots over the last few decades and building a model that translates between the two. So a system whose input is TSI and whose output is the Earth's surface temperature was studied, as the obvious and perhaps only possibility. The system is presumably linear for the small perturbations involved, and is presumably invariant (that is, does not change significantly with time), so the methods of Fourier analysis would seem to apply.

It was found empirically that the magnitude of the transfer function of this system had a prominent notch, no matter which datasets or periods were examined. The notch had a center frequency that corresponds to a period of ~11 years, the average length of the sunspot cycle but only half the length of a full solar cycle. The phases of the transfer function are unknown.

Thus we are interested in all possible systems that could explain a notch in the magnitude of a transfer function. We found that a 2nd order filter is the least complicated filter that could produce a notch, while higher order notch filters were just cascades of 2nd order notch filters. So, by Occam's razor, we focused just on a single 2nd order filter.

The transfer function of a 2nd order notch filter is a ratio of two 2nd order polynomials of frequency. Its numerator and denominator are each characterized by the roots of those polynomials, called the zeroes and poles respectively. Being 2nd order polynomials, there are two zeroes and two poles. To get a notch in the magnitude of the filter, the zeroes must be complex (not real) and conjugates of each other and the magnitude of the imaginary part must be greater than the magnitude of the real part, while the poles can be either a complex conjugate pair or both be real.

It is convenient to parameterize the zeroes and the poles in polar coordinates where angles are restricted to one quadrant (this also allows us to constrain the poles to be in the left half of the complex frequency plane, as required for system stability). To allow for any zeroes and poles meeting the above constraints (because we want to consider all possible 2nd order filters that produce notches), we need to allow for both positive and negative square roots of -1 (that is, $\pm i$) when factorizing each of the polynomials. The sign of i gives the sign of the phase changes produced by the transfer function, so this is expressing sign ambiguity in the phases. If we knew the phases of the empirical transfer function above we would know exactly which transfer function to consider, but we do not.

Hence, given any combination of two zeroes and two poles parameterized as above, there are in general four possible transfer functions. They were described above by two binary variables k and l, the sign signifiers, which respectively define the sign of the square roots of -1 in the numerator and denominator of the transfer function. Note that the process is that we start with zeroes and poles in one quadrant, *then* find the four associated transfer functions involving those poles and zeroes up to changes in sign of i. Of course, if we started with a given transfer function and factorized its two polynomials then it would have a unique combination of poles and zeroes.

Note also that the notion of a zero or pole requires a specification of the factorization variable—otherwise we would not be sure what it is that, when equal to the zero or pole, makes the polynomial's value zero. That specification is usually implicit. The factorization variable is usually a complex frequency, in which case the sign of the i in the complex frequency needs to be specified—because it is arbitrary, representing either sine or negative sine. Further, there is no particular reason the zeroes and poles have to be with respect to the same factorization variable, so we presumably have to allow them to differ. Thus there are a total of four combinations of factorization variables, different only in signs of i.

We calculated the step response of a general transfer function of a 2nd order filter from first principles: express the step input as a sum of sinusoids, note the effect of the transfer function on each input sinusoid to produce an output sinusoid at the same frequency, and sum the output sinusoids to form the step response. The calculation of the step response was an algebra-fest involving definite integrals from a reference work, but it was checked using numerical integration and then again by approximating the calculation using FFTs. In our examples we also confirmed the step response by checking it satisfied the linear differential equation from which the transfer function was derived.

Of the four possible transfer functions for a given combination of two zeroes and two poles, two have causal step responses and the other two have non-causal step responses. Clearly the causal step responses are possibilities for the Sun-Earth relationship, but what about the non-casual ones? Their non-causality dies out exponentially with decreasing time, so simply delaying the step response by a few years by combining the notch filter with a delay filter makes the step response of the combined filter causal, to a good approximation. So presumably the non-causal step responses are also possibilities for the Sun-Earth relationship, so long as they are combined with a delay.

As an example we looked at a simple RLC circuit that is known to act as a notch filter. We found its transfer function by solving the linear differential equation of the circuit when the input is a sinusoid. We then mapped that transfer function onto the form of the transfer function in our step-response calculation above, from which we obtained the step response—which was of course one of the causal ones (the circuit is real, so it is causal).

For a second example we flipped the sign of time in the circuit equation in the first example, to give a similar but crucially different transfer function. Its step response was one of the non-causal ones. By the way, because this simple series circuit is simpler than a general 2^{nd} order filter, i does not appear in the numerator of its transfer function when expressed in its simplest form, so there are only two versions of the transfer function to within the signs of i—and we did one example for each version.

Appendix A Special Functions

The following functions are used here but are not standard.

A.1 Indicator function

From the set of all propositions to 0 and 1:

$$I_{\text{proposition}} = \begin{cases} 1 & \text{the proposition is true} \\ 0 & \text{the proposition is false}. \end{cases} \tag{226}$$

For example, for some integer N,

$$5 + 3I_{N \text{ is even}} = \begin{cases} 8 & \text{if } N \text{ is even} \\ 5 & \text{if } N \text{ is odd}. \end{cases}$$

A.2 Signum Function

The signum function sgn (pronounced “signum”) gives the sign of its argument:

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases} \tag{227}$$

For example, $\text{sgn}(-0.4) = \text{sgn}(-10) = -1$ while $\text{sgn}(0.7) = \text{sgn}(24) = 1$.

A.3 Step Function

The (unit) step function switches from zero to one when its argument becomes positive:

$$\text{step}(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0. \end{cases} \tag{228}$$

For example, $\text{step}(-0.4) = \text{step}(-10) = 0$ while $\text{step}(0.7) = \text{step}(24) = 1$.

A.4 Eta Function

The eta function η (η is the Greek letter "eta") is useful for taking care of the inevitable factors of two that arise when dealing with sinusoids:

$$2^{\eta} = 2^{\eta(f)} = \begin{cases} 1 & \text{if } f = 0 \\ 2 & \text{if } f > 0. \end{cases} \tag{229}$$

η is simply the number of normal (that is, non-edge) frequencies in a context—here, because there is only one frequency variable and it is continuous, the only edge frequency is zero and η is either one or zero. We usually omit its frequency argument as understood and write "η" rather than "$\eta(f)$" in formulae.

A.5 Phase Function

Arctan needs extending to be able to compute polar-coordinate angles, for which we use the phase function pha (pronounced "far"). It gives the angle on a plane, in radians in $[0, 2\pi)$, that the point (x, y) makes with the x-axis:

$$\text{pha}(x, y) = \left[\tan^{-1}(y/x) + \pi I_{x<0}\right] \bmod 2\pi, \quad x, y \in \mathbb{R}. \tag{230}$$

If (x, y) is in the first quadrant, the phase function simplifies to

$$\text{pha}(x, y) = \tan^{-1}(y/x).$$

For example, $\text{pha}(1,0) = 0$, $\text{pha}(1,\sqrt{3}) = \pi/3$, $\text{pha}(0,1) = \pi/2$, and $\text{pha}(-1,0) = \pi$.

A similar function is the two-argument arctangent function atan2, but its range is $(-\pi, \pi]$.

Appendix B Acronyms

LIS	Linear invariant system
LPF	Low pass filter

Appendix C Electrical Engineering

Electrical engineers (EEs) have a lot of experience with systems, Fourier analysis, transfer functions, and step responses. This is the area of human endeavor that uses them explicitly and regularly, more than other areas. However EE's methodology and experience is focused on causal systems, because a circuit, by its very existence, is casual. Assuming causality makes solving circuits a lot simpler.

(By the way, the only method known to the author to find the step response of a 2nd order notch filter *without assuming causality* is using Fourier analysis as above, which is much more complicated than the methods routinely used by EEs. It was only using this method that the non-causal step responses came to light. Note that the 2nd order differential equation for the filter cannot be solved without first correctly guessing the form of the particular solution.)

The most common and powerful tool EEs use to solve circuits in the frequency domain is the Laplace transform. This is a generalization of the Fourier transform from sinusoids to exosoids (exponentially increasing or decreasing sinusoids, the product of an exponential growth factor and a sinusoid), but this requires that the transform be "one-sided", meaning times before zero are omitted from the integrals (because otherwise the integrand would include exposoids increasing without limit as time decreased). The Laplace transform of a function $g:\mathbb{R}\to\mathbb{R}$ is

$$\mathcal{L}\{g\} = L(s) = \int_0^\infty g(t)e^{-st}dt \tag{231}$$

where

$$s = \sigma + j\omega = \sigma + i\,2\pi f \tag{232}$$

is the complex frequency. The Laplace transform "assumes causality", in that it ignores what happens at negative times:

$$\mathcal{L}\{g\} = \mathcal{L}\{g(t)\} = \mathcal{L}\begin{Bmatrix} g(t) & t \geq 0 \\ \text{crazy}(t) & t < 0 \end{Bmatrix}. \tag{233}$$

In particular, the Laplace transform of a casual step response is identical to the Laplace transform of a non-causal step response that is the same for non-negative times.

Thus the Laplace transform cannot be used to detect non-causality. Instead we can use the Fourier transform because it is two sided—that is, takes function values over all time into account. Or we can solve circuit equations directly, but the circuit equations are linear differential equations and solving them is arduous—which is why EEs developed methods to avoid solving them directly from first principles.

Ask an EE website (such as this) or use a tool like Matlab to find a step response from the transfer function, circuit values, or the zeroes and poles, and you will only get the causal answers. These have been presumably calculated using formulas using the Laplace transform.

Digital circuits are relevant is no much as they approximate analog circuits, which are relevant because are described by the same simple linear differential equations as relationships often are in the natural world. However digital circuits are clocked, wherein the state of the circuit at each tick of the clock is computed from the state in the previous tick plus the change in input since the previous tick. With bazillions of ticks per second, such a circuit can be an excellent approximation of an analog filter. However, technically a digital circuit is not a LIS, because it is not invariant (though it is still linear). The output depends on when the input starts—any input that starts during the current clock cycle will produce the same output, because the circuit does nothing until the next cycle begins. Digital circuits are necessarily casual, because they start in a zeroed state and there can be no output until the input begins.

Acknowledgements

Thank you to Bernard Hutchins, an electrical engineer in Ithaca, New York, who persistently questioned my initial erroneous conclusion that the step response of a notch was necessarily non-causal [Hutchins, 2014]. He correctly pointed out that electronic notch filters can be causal, and included measurements of the step response of a circuit that implements a notch filter. (Initially, in the calculation of the step response of a 2nd order filter whose transfer function magnitude showed a notch, I overlooked the possibility of different signs of i in the transfer function. As it happens, I did the calculation with the sign signifiers k and l effectively both set to zero (Eq. (83))—leading to a non-causal step response only. So for a while it looked like a notch filter was non-causal, and necessarily must be accompanied by a delay to make it approximately causal. But once the role of the sign signifiers was realized, after review prompted by Bernie, it all made sense.)

References

Edminister, J. A. (1965). *Theory and Problems of Electric Circuits.* McGrawHill (Schaum's Outlines).

Evans, D. M. (2013, September 14). *The Optimal Fourier Transform (OFT).* Retrieved from http://jonova.s3.amazonaws.com/cfa/optimal-fourier-transform.pdf

Evans, D. M. (2016, January). The Notch-Delay Solar Theory.

Gradshteyn, I. S., & Ryzhik, I. M. (1980). *Table of Integrals, Series, and Products (Corrected and Enlarged Edition).* Academic Press.

Hutchins, B. A. (2014, July 30). *Application Note 413.* Retrieved from Electronotes: http://electronotes.netfirms.com/AN413.pdf

Okawa Electric Design. (2015). *RLC Band-stop Filter Design Tool*. Retrieved from Engineering Design Utilities: http://sim.okawa-denshi.jp/en/RLCtool.php

Tseng, Z. S. (2008). *Math 251 Class Notes.* Retrieved May 2015, 10, from Penn State University, Mathematics Department: http://www.math.psu.edu/tseng/class/Math251/Notes-2nd%20order%20ODE%20pt2.pdf